A brief introduction to renormalization group

Yi Zhou

yizhou@zju.edu.cn

Zhejiang University, Hangzhou

Nanjing, August 28, 2016

Contents

Why renormalization group (RG)?

- RG plays an essential role in modern physics, including quantum field theory, statistical mechanics, condensed matter physics, and even dissipative quantum tunnelling.

- RG is not only a theoretical method, but also a morden way to think physics.

Brief history

- Quantum electrodynamics (QED): A quantum field theory (QFT)
  - The problem of infinities: Ultraviolet (UV) divergence Weisskopf
  - Renormalization: Feynman, Schwinger, Tomonaga, Dyson, *et al.*
  - QFT and RG: Peterman & Strückberg, Gell-Mann & Low, Bogoliubov & Shirkov
    - Consider QED in the limit of photons and massless electrons.
    - In a massive theory, the renormalized charge can be defined through the electric interaction of particles at rest, say, Coulomb interaction.
    - This definition is no longer applicable to massless particles, which always travel at the speed of light.
    - Then it is necessary to introduce an arbitary mass or energy scale μ to define the renormalized charge e.

  * This effective charge then depends on μ. The set of transformations of physical constants accociated with the change in scale μ is called *renormalization group.*

- Standard model: Renormalizability becomes a criterion for a QFT

Brief history

- Critical phenomena: Other infinities (Infrared (IR) divengence)
  - Phases and phase transition
  - Mean field theory
  - Fluctuations
  - The non-decoupling of scales
  - The cutoff as scale paramter

The cutoff as scale parameter: Effective field theory at criticality

To ensure that we remain in the low-energy domain, we would like to take the cutoff to be infinite, but this cannot be done by declaration. In the absence of external fields, the action of the system does not contain an intrinsic energy scale apart from the cutoff. Thus, the cutoff disappears from the action when we reduce all quantities to dimensionless form. The only way to tell whether it is finite or infinite is to calculate some physical quantity with dimension, such as the correlation length, from the theory. The cutoff is infinite when the correlation length diverges, in which case the system is said to be at a critical point. To approach the limit of infinite cutoff, therefore, we must adjust the parameters so as to make the system "go critical".

$$\langle \phi(x)\phi(y)\rangle \underset{|x-y|\to\infty}{\longrightarrow} Ce^{-|x-y|/\tilde{\xi}}$$

Here the correlation length $\tilde{\xi}$ is measured in the same unit as x . Using Λ^{-1} as unit for distance, we have

$$\frac{|x-y|}{\tilde{\xi}} = \frac{|x-y|\,\Lambda}{\xi}$$

where ξ is dimensionless:

$$\xi = \Lambda\tilde{\xi}.$$

Ignoring the pathological case $\tilde{\xi} = 0$, we see that an infinite cutoff corresponds to the limit $\xi \to \infty$.

Brief history

- Kadanoff and Wilson's renormalization group
  - Scaling hypothesis
  - Continuum limit
  - Gaussian fixed point and mean field theory
  - Go beyond Guassian fixed point

Brief history

- The concept of effective field theory
  - More is different
  - Emergent phenomena: Different physical laws will appear at different scales
  - Renormalizability: A criterion to construct effective field theories

Critical phenomena

Critical exponents

For continuous phase transitions, the critical exponents will not vary along the phase boundary, which characterize the universality class for phase transitions.

The critical exponents for ferromagnetic systems are list in the following table, where $t = (T - T_c)/T_0, h = (H - H_c)/H_0$, T_0 and H_0 are nonuniversal constants.

critical exponent	associated quantity	singular part
α	heat capacity	$C_H \propto \|t\|^{\alpha}$
β	magnetization	$m \propto \|t\|^{\beta}$
γ	susceptibility	$\chi \propto \|t\|^{-\gamma}$
δ	magnetization	$m \propto \|h\|^{1/\delta}$
η	correlation function	$G(\vec{r}) \propto \frac{e^{-r/\xi}}{\|r\|^{d-2+\eta}}$
ν	correlation length	$\xi \propto \|t\|^{-\nu}$

Critical phenomena

Scaling hypothesis

Widom made the following homogeneity assumption for the singular form of free energy in the vicinity of a phase transition,

$$f(t,h) = t^{2-\alpha} g_f\left(\frac{h}{t^{\Delta}}\right),$$

where g_f is a scaling funciton for free energy (which is different for $t > 0$ and $t < 0$), and the exponents α and Δ depend on the critical point under consideration.

With the help of this form of free energy, one can determine a series of critical exponents using the two independent exponents α and Δ.

- Heat capacity:

The internal energy is given by

$$E \propto \frac{\partial f}{\partial t} = (2-\alpha)t^{1-\alpha} g_f\left(\frac{h}{t^{\Delta}}\right) - \Delta h t^{1-\alpha-\Delta} g_f'\left(\frac{h}{t^{\Delta}}\right) \equiv t^{1-\alpha} g_E\left(\frac{h}{t^{\Delta}}\right),$$

and the heat capacity reads

$$C_H \propto t^{-\alpha} g_C\left(\frac{h}{t^{\Delta}}\right),$$

where $g_C\left(\frac{h}{t^{\Delta}}\right) = (1-\alpha) g_E\left(\frac{h}{t^{\Delta}}\right) - \Delta g_E'\left(\frac{h}{t^{\Delta}}\right)$.

- Magnetization:

$$m(t,h) = -\frac{\partial f}{\partial h} \propto -t^{2-\alpha-\Delta} g_f' \left(\frac{h}{t^\Delta}\right) \equiv t^{2-\alpha-\Delta} g_m \left(\frac{h}{t^\Delta}\right)$$

In the limit $x \to 0$, $g_m(x)$ is a constant, so that

$$m(t, h=0) \propto t^{2-\alpha-\Delta} g_m(0),$$

i.e., $\beta = 2 - \alpha - \Delta$.

On the other hand, if $x \to \infty$, $g_m(x) \propto x^p$, we must have $p\Delta = 2 - \alpha - \Delta$. Hence

$$m(t=0, h) \propto h^p = h^{\frac{2-\alpha-\Delta}{\Delta}},$$

i.e., $\delta = \Delta/(2 - \alpha - \Delta) = \Delta/\beta$.

- Susceptibility:

$$\chi(t,h) \propto t^{2-\alpha-2\Delta} g_\chi\left(\frac{h}{t^\Delta}\right)$$

So that

$$\chi(t,h=0) \propto t^{2-\alpha-2\Delta} g_\chi(0)\,,$$

i.e., $\gamma = 2\Delta + \alpha - 2$.

A number of exponent identities follows, for example,

$$\alpha + 2\beta + \gamma = 2,$$

and

$$\delta - 1 = \gamma/\beta.$$

- Correlation length:

Close to criticality, the correlation length ξ is the most imporant length scale, and is solely responsible for singular contribution to thermodynamic quantities. Since $\ln Z$ is dimensionless and extensive,

$$\ln Z = \frac{L^d}{\xi^d} g_s + \frac{L^d}{a^d} g_a.$$

From the assumption of the sigular form for $f(t,h)$, we have

$$\frac{g_s}{\xi^d} = t^{2-\alpha} g_f,$$

and

$$\xi(t,h) \propto t^{\frac{\alpha-2}{d}} g_\xi(\frac{h}{t^\Delta}),$$

i.e.,

$$2 - \alpha = d\nu.$$

- Correlation functions:

At the critical point,

$$G(\vec{r}) \propto \frac{1}{r^{d-2+\eta}}.$$

On the other hand

$$\chi \propto \int d^d\vec{r} G(\vec{r}) \propto \int^{\xi} d^d\vec{r} \frac{1}{r^{d-2+\eta}} \propto \xi^{2-\eta} \propto t^{-\nu(2-\eta)}.$$

So that

$$\gamma = (2-\eta)\nu.$$

The critical system has an additional dilation symmetry,

$$G_{critical}(\lambda\vec{r}) = \lambda^p G_{critical}(\vec{r}),$$

which implies scale invariance or self-similarity.

It is not in general possible to see directly how such a symmetry requrirment constrains the effective Hamiltonian. We shall instead precrible a less direct route by following the effects of the dilation operation on the effective energy, namely, a process known as "*renormalization group*" .

- **Homework**: Solve one-dimensional Ising model using transfer matrix method. Find out the transition temperature T_c, and compute the critical exponents α, β, and γ.

Framework: General theory and concepts

Suppose we have an effective field theory defined thorugh the action,

$$S[\phi] = \sum_a g_a \mathcal{O}_a[\phi],$$

where ϕ is some field, g_a are coupling constants and $\mathcal{O}_a[\phi]$ a certain set of operators. "Renormalization" is a scheme to derive a set of Gell-Mann-Low equations describing the change of the coupling constants $\{g_a\}$ as fast fluctuation modes of the theory are sucessively integrated out.

There are a number of methodologically different procedures by which the set of flow equations can be obtained from the microscopic theory. In general, RG transformations contains the following three stages.

- **Subdivision of the field manifold**
- **RG transformation**
- **Rescaling**

Three stages of RG

- **Subdivision of the field manifold**, the practice may be different in different situations, for instance

  - We may proceed a generalized block spin scheme, and integrate over all degrees of freedom within a certain structural unit in real space.

  - We may separate fast and slow field modes ($\phi_>$ and $\phi_<$) in momentum space, and integrate over a shell of $\Lambda/b \leq |\vec{p}| < \Lambda$, where Λ is the momentum cutoff and $b > 1$.

  - We may decide to integrate over all the high-lying degrees of freedom $\lambda^{-1} \leq |\vec{p}|$. In this case, the encountered divergent integrals may be handled with dimensional regularization, for example.

  - We may use the short-distance real space cutoff underlying the socalled *operator product expansion* (OPE).

- **RG transformation**, the central part of RG, is to actually integrate over short-range fluctuations.

  - For RG in momentum space, a frequently used approximation scheme is the *loop expansion*.

  - Following the procedure, an integration over the fast degrees of freedom gives rise to an action

$$S'[\phi_<] = \sum_a g'_a \mathcal{O}'_a[\phi_<],$$

    where coupling constants of the remaining slow fields are altered. Notice that the integration over fast field fluctuations may lead to the generation of "new" operators.

- **Rescaling**, to recover the form of $S[\phi]$ with the same set of $\{\mathcal{O}_a[\phi]\}$.

  - One then rescales momentum/frequency so that the rescaled field ϕ' fluctuates on the same scales as the original field ϕ, i.e., one sets

    $$\boxed{q \to bq, \omega \to b^z \omega,}$$

    where the dynamical exponent z depends on the effective dispersion.

  - We will also designate a dimension L^{d_ϕ} to for the field ϕ, so as to compensate for the factor b^x arising after the renormalization of the operator. The rescaling

    $$\boxed{\phi \to b^{d_\phi} \phi}$$

    is known as field renormalization. It renders the "leading" operator in the action scale invarinat.

  - As a result of all these manipulations, we obtain a renormalized action

    $$S[\phi] = \sum_a g'_a \mathcal{O}_a[\phi],$$

which is entirely described by the set of changed coupling contants, i.e. the effect of the RG transformation is fully encapsulated in the mapping

$$\mathbf{g}' = \tilde{\mathbf{R}}(\mathbf{g}).$$

– Letting $l \equiv \ln b$, we have the **Gell-Mann-Low equation**

$$\frac{d\mathbf{g}}{dl} = \mathbf{R}(\mathbf{g}),$$

where $\mathbf{R}(\mathbf{g}) = \lim_{l\to 0}(\tilde{\mathbf{R}}(\mathbf{g}) - \mathbf{g})/l$ is the generalized β-function.

Framework: General theory and concepts

Analysis of the Gell-Mann-Low equation

The Gell-Mann-Low equation represents the principle result of an RG analysis.

- **Fixed points**, where the RG flow becomes stationary,

$$\mathbf{R}(\mathbf{g}^*) = 0.$$

  - At a fixed point, the system becomes self-similar, namely, does not change under rescaling.
  - One can analyze the RG flow close to a fixed point $\mathbf{g}^*$ by expanding

$$\mathbf{R}(\mathbf{g}) \simeq W(\mathbf{g} - \mathbf{g}^*),$$

    where the matrix W is given by

$$W_{ab} = \frac{\partial R_a}{\partial g_b}\Big|_{\mathbf{g}=\mathbf{g}^*}.$$

Then the matrix W can be diagonalized as follows,

$$\phi_\alpha^T W = \lambda_\alpha \phi_\alpha^T.$$

- **Scaling field.** Defining

$$v_\alpha = \phi_\alpha^T \cdot (\mathbf{g} - \mathbf{g}^*),$$

we have

$$\frac{dv_\alpha}{dl} = \phi_\alpha^T \cdot \frac{d\mathbf{g}}{dl} = \phi_\alpha^T \cdot \mathbf{R}(\mathbf{g}) = \phi_\alpha^T W(\mathbf{g} - \mathbf{g}^*) = \lambda_\alpha \phi_\alpha^T \cdot (\mathbf{g} - \mathbf{g}^*) = \lambda_\alpha v_\alpha.$$

Under renormalization, the coefficients v_α change by a mere scaling factor λ_α, thereby they are called *scaling fields*.

  – Relevant scaling field. For $\lambda_\alpha > 0$, the flow is directly away from the fixed point. The associated scaling field v_α is said to be relevant, which indicates some instability.

  – Irrelevant scaling field. For $\lambda_\alpha < 0$, the flow is attracted by the fixed point. The associated scaling field is said to be irrelevant.

  – Marginal scaling field. For $\lambda_\alpha = 0$, the scaling field is unchanged under the flow, are termed marginal.

- **Different types of fixed points.**
  - **Stable fixed point.** All the scaling fields are irrelevant or, at worst, marginal. These points define what we might call "stable phase of matter".
  - **Unstable fixed point.** Complementary to the stable fixed points, there are unstable fixed points, whose scalling fields are all relevant.
  - There is the gneric class of fixed points with both relevant and irrelevant scaling fields. These points are of particular interest because they can be associated with **phase transition**.

Example 1: Warm up with Gaussian model

Simplest model: Gaussian model

We will take free massless scalar field in D-dimension as the first example, which represents the gapless Gaussian fluctuations in the ordered side in Landau's theory for continuous phase transition. The simplest model of such Gaussian fluctuations has the following action,

$$
\begin{aligned}
S &= \frac{1}{2}\int d^D\vec{r}\,(\nabla\phi)^2 \\
&= -\frac{1}{2}\int \frac{d^D\vec{p}}{(2\pi)^D}\phi^*(\vec{p})p^2\phi(\vec{p}).
\end{aligned}
$$

- **Subdivision of fields**

Firstly separate the scalar field ϕ with a cut-off Λ into two parts, slow varying part with smaller cut-off $\tilde{\Lambda} < \Lambda$ and rapid varying part. Thus we have

$$\begin{aligned}
\phi(\vec{r}) &= \phi_<(\vec{r}) + \phi_>(\vec{r}), \\
\phi_<(\vec{r}) &= \int_{0\le|p|<\tilde{\Lambda}} \frac{d^D\vec{p}}{(2\pi)^D} e^{i\vec{p}\cdot\vec{r}} \phi(\vec{p}), \\
\phi_>(\vec{r}) &= \int_{\tilde{\Lambda}\le|p|<\Lambda} \frac{d^D\vec{p}}{(2\pi)^D} e^{i\vec{p}\cdot\vec{r}} \phi(\vec{p}).
\end{aligned}$$

Substitute the above into the action, we obtain that

$$\begin{aligned}
S[\phi] &= \frac{1}{2}\int d^D\vec{r}\left[(\nabla\phi_<)^2 + (\nabla\phi_>)^2\right] \\
&= -\frac{1}{2}\int_{0\le|p|<\tilde{\Lambda}} \frac{d^D\vec{p}}{(2\pi)^D}\phi^*(\vec{p})p^2\phi(\vec{p}) - \frac{1}{2}\int_{\tilde{\Lambda}\le|p|<\Lambda} \frac{d^D\vec{p}}{(2\pi)^D}\phi^*(\vec{p})p^2\phi(\vec{p}).
\end{aligned}$$

- **RG transformation**

The next step is to integrate out the rapid varying section $\phi_>(\vec{r})$, we have

$$\begin{aligned} S'[\phi_<] &= \frac{1}{2}\int d^D\vec{r}\,(\nabla\phi_<)^2 \\ &= -\frac{1}{2}\int_{\underline{0\le|p|<\tilde{\Lambda}}} \frac{d^D\vec{p}}{(2\pi)^D}\phi^*(\vec{p})\vec{p}^2\phi(\vec{p}). \end{aligned}$$

Note that $S'[\phi]$ is not of the same form as $S[\phi]$. We need to do rescaling to recover the form.

- **Rescaling**

The third step is to rescale $\phi_<$, to do this, we define

$$\vec{p}' = \frac{\Lambda}{\tilde{\Lambda}}\vec{p} = b\vec{p},$$

and new fields

$$\phi'(\vec{p}') = \zeta^{-1}\phi(\vec{p}'/b) = \zeta^{-1}\phi(\vec{p}),$$

and choose ζ such that a certain coupling in the quadratic part of the action has a fixed coefficient. Since

$$\begin{aligned} S'[\phi_<] &= -\frac{1}{2}\int_{0\leq|p|<\tilde{\Lambda}} \frac{d^D\vec{p}}{(2\pi)^D}\phi^*(\vec{p})p^2\phi(\vec{p}) \\ &= -\frac{1}{2}\int_{0\leq|p'|<\Lambda} \zeta^2 b^{-D-2}\frac{d^D\vec{p}'}{(2\pi)^D}\phi'^*(\vec{p}')p'^2\phi'(\vec{p}'). \end{aligned}$$

we will choose

$$\boxed{\zeta = b^{1+\frac{D}{2}}.}$$

So that

$$S'[\phi'] = S[\phi].$$

It means that the Gaussian action does not change under the RG transformation. And the free Gaussian field theory itself is a fixed point.

Example 1: Warm up with Gaussian model

Quadratic perturbations: Relevant, irrelevant and marginal operators

Having found the Gaussian fixed point, we next classify its perturbations as relevant, irrelevant or marginal.

Let us start with quadratic perturbation,

$$\delta S = -\frac{1}{2}\int_{0\leq|p|<\Lambda}\frac{d^D\vec{p}}{(2\pi)^D}\phi^*(\vec{p})r(p)\phi(\vec{p}),$$

where the coupling function $r(p)$ is assumed to have the Taylor expansion,

$$r(p) = r_0 + r_2p^2 + r_4p^4 + \cdots .$$

One also often writes

$$r_0 = m_0^2,$$

and refers to m_0^2 as the mass term. Since

$$\begin{aligned}
\delta S'[\phi_<] &= -\frac{1}{2}\int_{0\leq|p|<\tilde{\Lambda}} \frac{d^D\vec{p}}{(2\pi)^D}\phi^*(\vec{p})r(p)\phi(\vec{p}) \\
&= -\frac{1}{2}\zeta^2 b^{-D}\int_{0\leq|p'|<\Lambda} \frac{d^D\vec{p}'}{(2\pi)^D}\phi'^*(\vec{p}')r(p'/b)\phi'(\vec{p}') \\
&= -\frac{1}{2}b^2\int_{0\leq|p'|<\Lambda} \frac{d^D\vec{p}'}{(2\pi)^D}\phi'^*(\vec{p}')r(p'/b)\phi'(\vec{p}') \\
&= -\frac{1}{2}\int_{0\leq|p'|<\Lambda} \frac{d^D\vec{p}'}{(2\pi)^D}\phi'^*(\vec{p}')r'(p')\phi'(\vec{p}'),
\end{aligned}$$

we have, by comparison,

$$r'(p') = b^2 r(p'/b).$$

So that the Taylor coefficients obey that

$$\boxed{\begin{aligned}
r'_0 &= b^2 r_0, \\
r'_2 &= r_2, \\
r'_4 &= b^{-2} r_4,
\end{aligned}}$$

and so on. Thus, we find that r_0 is relevant, r_4' is irrelevant, and r_2 is marginal.

- This is a concrete example of how in the low energy physics the couling function $r(k)$ reduces to a few coupling constants.

Example 1: Warm up with Gaussian model

Revisit scaling hypothesis and critical exponents

Without the loss generality, we may assume that t and h rescale as

$$\begin{aligned} t' &= b^{y_t} t, \\ h' &= b^{y_h} h, \end{aligned}$$

by the RG flow. Then the free energy density will be rescaled as

$$f'(t', h') = b^{-D} f(b^{y_t} t, b^{y_h} h).$$

Fixing $b^{y_t} t = c$ (can be any positive constant), one has

$$\begin{aligned} b &= c^{1/y_t} t^{-1/y_t}, \\ b^{y_h} h &= c^{y_h/y_t} h / t^{y_h/y_t}. \end{aligned}$$

Thus $f(t, h)$ must be scaled as

$$f = t^{D/y_t} g_f \left(\frac{h}{t^{y_h/y_t}} \right).$$

We have

$$\begin{aligned}\alpha &= 2 - D/y_t,\\ \Delta &= y_h/y_t.\end{aligned}$$

The remaining thing is to determin y_t and y_h, which can be computed from different models and by different RG approaches. We shall compute y_t and y_h with the Gaussian model and its perturbation (for instance, ϕ^4 model).

In order to do this, let us recall the Landau function for ferromagnetic transition, which can be expanded to the fourth order of the order paramter ϕ,

$$\mathcal{L}(\phi, h, t) = \sum_{n=0}^{4} a_n(h,t)\phi^n,$$

where $a_n(h,t)$ can be written as

$$a_n(h,t) = b_n + c_n h + d_n t.$$

By the symmetry argument, the most general form for Landau function near the critical point is

$$\mathcal{L}(\phi, h, t) = c_1 h\phi + d_2 t\phi^2 + c_3 h\phi^3 + b_4\phi^4,$$

where $d_2 > 0$ and $b_4 > 0$ for stability.

At the level of the Gaussian model, we may neglect ϕ^3 and ϕ^4 terms at first, and only consider the case of $t > 0$. In this case, t is nothing but the mass term r_0 in the Gaussian model. So that

$$y_t = 2.$$

For the linear term of ϕ, it is easy to find out that

$$h' = \zeta h = b^{1+\frac{D}{2}} h,$$

therefore,

$$y_h = 1 + \frac{D}{2}.$$

- Homework: Compute critical exponents $\alpha, \beta, \gamma, \delta, \eta, \gamma$ for the Gaussian model.

Example 2: ϕ^4 theory

Quartic perturbation to Gaussian model: The ϕ^4 theory

Now we consider the quartic perturbation to the Gaussian model

$$\begin{aligned}\delta S &= \frac{1}{4!}\int_{|k|<\Lambda}\prod_{i=1}^{4}\frac{d^D\vec{k}_i}{(2\pi)^D}u(\vec{k}_4\vec{k}_3\vec{k}_2\vec{k}_1)\phi^*(\vec{k}_4)\phi^*(\vec{k}_3)\phi(\vec{k}_2)\phi(\vec{k}_1)\delta^D(\vec{k}_4+\vec{k}_3-\vec{k}_2-\vec{k}_1)\\ &\equiv \frac{1}{4!}\int_{|k|<\Lambda}u(4321)\phi^*(4)\phi^*(3)\phi(2)\phi(1),\end{aligned}$$

where the coupling function obeys the symmetry condition

$$u(4321) = u(3421) = u(4312).$$

The RG transformation can be done through

$$e^{-S'[\phi_<]} = e^{-S_0[\phi_<]} \left\langle e^{-\delta S[\phi_<,\phi_>]} \right\rangle_{0>} \equiv e^{-S_0 - \delta S'}.$$

Linked expansion will give rise to

$$\delta S' = \langle \delta S \rangle - \frac{1}{2} \left(\left\langle \delta S^2 \right\rangle - \langle \delta S \rangle^2 \right) + \cdots .$$

Since $\delta S \propto u$, the above is a perturbation expansion and can be evaluated through Feynman diagrams.

Linked expansion

$$\ln \left\langle e^{-U} \right\rangle = - \langle U \rangle + \frac{1}{2} \left\langle U^2 \right\rangle_c + \cdots + \frac{(-1)^n}{n!} \langle U^n \rangle_c ,$$

where $\langle \rangle_c$ denotes connected average.

Tree-level RG for ϕ^4 theory

The leading term has the form

$$\langle \delta S \rangle = \frac{1}{4!} \left\langle \int_{|k|<\Lambda} u(4321)\, (\phi_< + \phi_>)^*_4\, (\phi_< + \phi_>)^*_3\, (\phi_< + \phi_>)_2\, (\phi_< + \phi_>)_1 \right\rangle_{0>} .$$

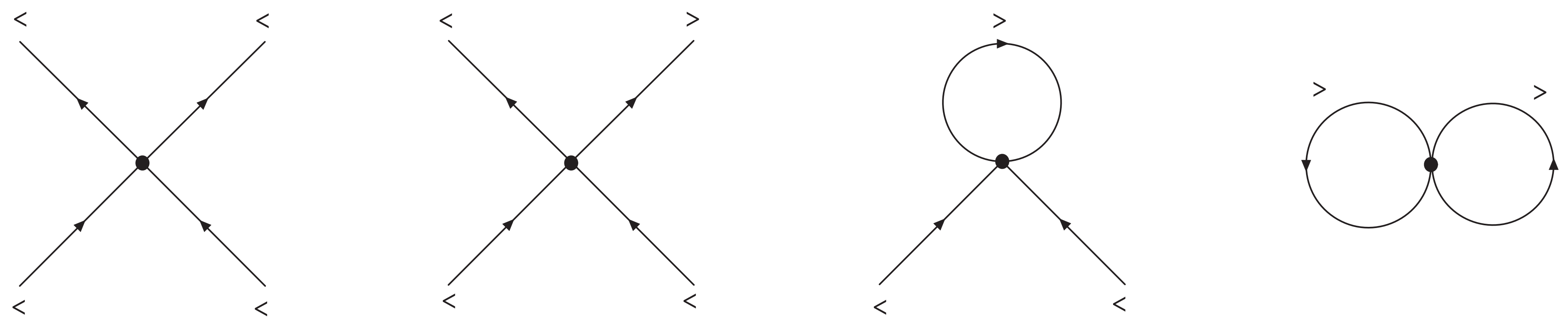
There are 16 possible monomials that fall into four sets.

8 terms with odd number of fast $\phi_>$ modes, vanish by symmetry

1 term with all fast $\phi_>$ modes, make a constant contribution, independent of $\phi_<$

1 term with all slow $\phi_<$ modes, tree-level

6 terms with two fast $\phi_>$ and two slow $\phi_<$ modes.



Consider the third set with all slow modes, say, the tree-level terms in field theory, and rewrite it in terms of new momenta and fields, we find that

$$\begin{aligned}
\delta S'_{4,tree} &= \frac{1}{4!}\int_{|k|<\tilde{\Lambda}}\prod_{i=1}^{4}\frac{d^D\vec{k}_i}{(2\pi)^D}u(\vec{k}_4\vec{k}_3\vec{k}_2\vec{k}_1)\phi^*(\vec{k}_4)\phi^*(\vec{k}_3)\phi(\vec{k}_2)\phi(\vec{k}_1)\delta^D(\vec{k}_4+\vec{k}_3-\vec{k}_2-\vec{k}_1) \\
&= \frac{1}{4!}\zeta^4 b^{-3D}\int_{|k|<\Lambda}\prod_{i=1}^{4}\frac{d^D\vec{k}'_i}{(2\pi)^D}u(\vec{k}'_4/b,\vec{k}'_3/b,\vec{k}'_2/b,\vec{k}'_1/b)\phi'^*(\vec{k}'_4)\phi'^*(\vec{k}'_3)\phi'(\vec{k}'_2)\phi'(\vec{k}'_1) \\
&\quad\times\delta^D(\vec{k}'_4+\vec{k}'_3-\vec{k}'_2-\vec{k}'_1) \\
&= \frac{1}{4!}b^{4-D}\int_{|k|<\Lambda}\prod_{i=1}^{4}\frac{d^D\vec{k}'_i}{(2\pi)^D}u(\vec{k}'_4/b,\vec{k}'_3/b,\vec{k}'_2/b,\vec{k}'_1/b)\phi'^*(\vec{k}'_4)\phi'^*(\vec{k}'_3)\phi'(\vec{k}'_2)\phi'(\vec{k}'_1) \\
&\quad\times\delta^D(\vec{k}'_4+\vec{k}'_3-\vec{k}'_2-\vec{k}'_1).
\end{aligned}$$

So that

$$b^{4-D}u(\vec{k}'_4/b,\vec{k}'_3/b,\vec{k}'_2/b,\vec{k}'_1/b) = u'(\vec{k}'_4,\vec{k}'_3,\vec{k}'_2,\vec{k}'_1).$$

Carrying out the Taylor expansion,

$$u = u_0 + u_1 k + u_2 k^2 + \cdots ,$$

we see that

$$\boxed{\begin{array}{rcl} u'_0 &=& b^{4-D} u_0, \\ u'_1 &=& b^{3-D} u_1, \\ u'_2 &=& b^{2-D} u_2, \end{array}}$$

and so on and so forth. It is clear that when $D = 4$, only u_0 is marginal and other terms are irrelevant.

This is the reason why the ϕ^4 theory in $D = 4$ is described by a coupling constant but not a coupling function.

For $D > 4$, all the ϕ^4 terms are irrelevant.

- **Homework**: Analyze the relevance of operators ϕ^n, $n > 4$ in different dimensions D.

One-loop RG for ϕ^4 theory

According to previous tree-level analysis, higher gradient terms are irrelevant, we consider the following simplified model, in D-dimension,

$$S[\phi] = \int d^D\vec{r}\left[\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}r\phi^2 + \frac{\lambda}{4!}\phi^4 - h\phi\right],$$

where h is the external magnetic field.

There are three coupling constants, r, λ and h, in the simplified ϕ^4 theory.

- **Subdivision of fields**

By seperating $\phi = \phi_< + \phi_>$, up to the one-loop, we have

$$S[\phi] = S_<[\phi_<] + S_>[\phi_>] + S_I[\phi_<, \phi_>],$$

with

$$\begin{aligned} S_<[\phi_<] &= \int d^D\vec{r}\left[\frac{1}{2}(\nabla\phi_<)^2 + \frac{r}{2}\phi_<^2 + \frac{\lambda}{4!}\phi_<^4 - h\phi_<\right], \\ S_>[\phi_>] &= \int d^D\vec{r}\left[\frac{1}{2}(\nabla\phi_>)^2 + \frac{r}{2}\phi_>^2\right], \\ S_I[\phi_<, \phi_>] &= \frac{\lambda}{4}\int d^D\vec{r}\left[\phi_<^2\phi_>^2 + \cdots\right], \end{aligned}$$

where we only keep the $\phi_<^2\phi_>^2$ term at one-loop level. (It is also easy to verify that other terms will lead to two-loop contribution at leading order.) Thus, the effective action is given by the following connected diagrams contribution

$$S'[\phi_<] = S_<[\phi_<] + \langle S_I[\phi_<, \phi_>]\rangle_> - \frac{1}{2}\left\langle S_I[\phi_<, \phi_>]^2\right\rangle_{>c}.$$

- **RG transformation & resacaling**

The first order average reads

$$\langle S_I[\phi_<, \phi_>]\rangle_> = \frac{\lambda}{4}\int_> \frac{d^D\vec{p}'}{(2\pi)^D}\frac{1}{r+p'^2}\int_< \frac{d^D\vec{p}}{(2\pi)^D}\phi^*_<(\vec{p})\phi_<(\vec{p}).$$

The integral can be computed by expanding r to the first order,

$$\int_> \frac{d^D\vec{p}'}{(2\pi)^D}\frac{1}{r+p'^2} = I_1 - rI_2,$$

with

$$I_\alpha = \int_> \frac{d^D\vec{p}'}{(2\pi)^D}\frac{1}{p'^{2\alpha}} = \Omega_D\Lambda^{D-2\alpha}\int_{b^{-1}}^1 dp p^{D-1-2\alpha} = \frac{\Omega_D\Lambda^{D-2\alpha}}{D-2\alpha}(1-b^{2\alpha-D}),$$

where

$$\Omega_D = \frac{(2\pi)^{\frac{D}{2}}}{\Gamma(\frac{D}{2})(2\pi)^D}$$

is the volume of the D-dimensional unit sphere.

So that by the momenta and field scaling,

$$S^{(2)}[\phi'] = \frac{b^2}{2}\left[r + \frac{\lambda\Omega_D}{2(D-2)}(1-b^{2-D}) - \frac{r\lambda\Omega_D}{2(D-4)}(1-b^{4-D})\right]\int d^D\vec{r}\phi'^2.$$

The second order will lead to the contribution proportional to $\phi_<^4$,

$$\frac{1}{2}\left\langle S_I[\phi_<,\phi_>]^2\right\rangle_{>c} \simeq \frac{\lambda^2}{16}\int d^D\vec{r}\phi_<^4 \int_> \frac{d^D\vec{q}}{(2\pi)^D}\frac{1}{(r+q^2)^2} = \frac{\lambda^2 I_2}{16}\int d^D\vec{r}\phi_<^4 + O(\lambda^2 r).$$

Evaluating the integral and rescaling, we have the $S^{(4)}[\phi']$ as follows,

$$S^{(4)}[\phi'] = b^{4-D}\left(\frac{\lambda}{4!} - \frac{\lambda^2\Omega_D}{16}\frac{1-b^{4-D}}{D-4}\right)\int d^D\vec{r}\phi'^4.$$

Finally, the linear term after rescaling reads

$$S^{(1)}[\phi'] = hb^{1+\frac{D}{2}}\int d^D\vec{r}\phi'.$$

Thus, to the one-loop order, the coupling constants rescale as follows,

$$
\begin{aligned}
r &\to b^2\left[r+\frac{\lambda\Omega_D}{2(D-2)}(1-b^{2-D})-\frac{r\lambda\Omega_D}{2(D-4)}(1-b^{4-D})\right],\\
\lambda &\to b^{4-D}\left(\lambda-\frac{3\lambda^2\Omega_D}{2}\frac{1-b^{4-D}}{D-4}\right),\\
h &\to hb^{1+\frac{D}{2}}.
\end{aligned}
$$

- **ϵ-expansion:**

It is interesting that $\epsilon = 4 - D$ can serves an expansion paramter. We set $D = 4 - \epsilon$ and evaluate the above to leading order of ϵ. Note that $\Omega_{4-\epsilon} \simeq \Omega_4 = \frac{1}{8\pi^2}$. Therefore,

$$\begin{aligned}
r &\rightarrow b^2 \left[r + \frac{\lambda}{32\pi^2}(1 - b^{-2}) - \frac{r\lambda}{16\pi^2} \ln b \right], \\
\lambda &\rightarrow (1 + \epsilon \ln b) \left(\lambda - \frac{3\lambda^2}{16\pi^2} \ln b \right), \\
h &\rightarrow h b^{3-\epsilon/2}.
\end{aligned}$$

- **Gell-Mann-Low equations**

Setting $b = e^l$, we have the Gell-Mann-Low equations

$$\begin{aligned} \frac{dr}{dl} &= 2r + \frac{\lambda}{16\pi^2} - \frac{r\lambda}{8\pi^2}, \\ \frac{d\lambda}{dl} &= \epsilon\lambda - \frac{3\lambda^2}{16\pi^2}, \\ \frac{dh}{dl} &= \frac{6-\epsilon}{2}h. \end{aligned}$$

We find that one-loop correction will not affect the scaling behavior of h. We shall consider the simple case with $h = 0$ and analyze the RG flow for r and λ at first.

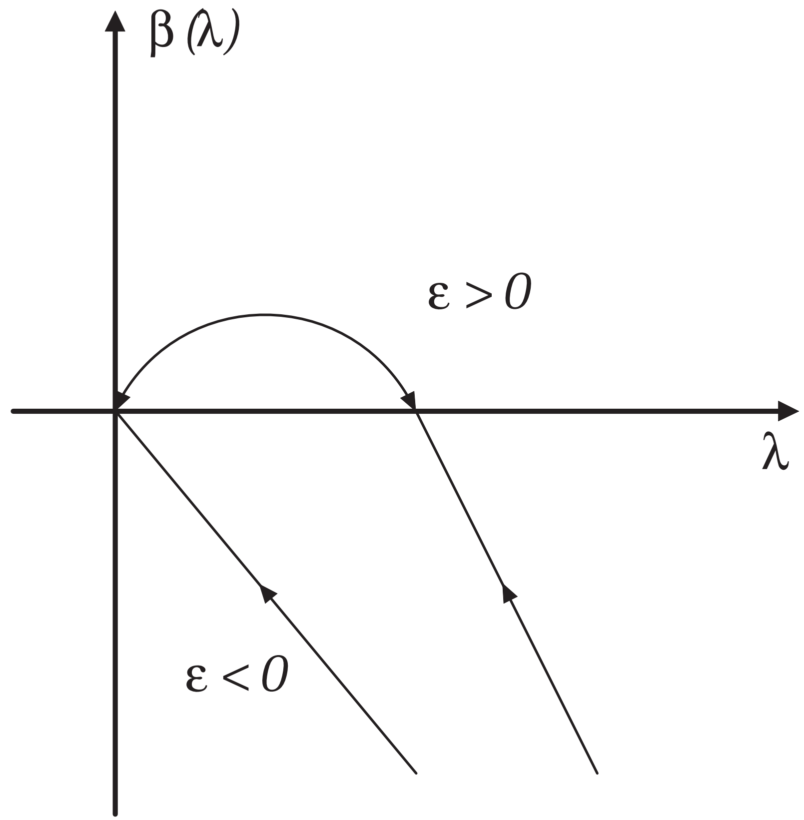
The β-function

$$\beta(\lambda) = \epsilon\lambda - \frac{3\lambda^2}{16\pi^2}$$

exhibits different behaviors for $\epsilon > 0$ and $\epsilon < 0$.

When $\epsilon < 0$, there is only one trivial fixed point.

While for $\epsilon > 0$, there exists a non-trivial fixed point.


β(λ)
ε > 0
λ
ε < 0

- **Fixed point:**

  The fixed points are given by

$$\begin{aligned} 2r + \frac{\lambda}{16\pi^2} - \frac{r\lambda}{8\pi^2} &= 0, \\ \epsilon\lambda - \frac{3\lambda^2}{16\pi^2} &= 0. \end{aligned}$$

  - **Gaussian fixed point:** a trival fixed point is Gaussian fixed point given by

$$r^* = \lambda^* = 0.$$

    The W-matrix is

$$W = \begin{pmatrix} 2 & \\ & \epsilon \end{pmatrix}.$$

  - **Non-Gaussian fixed point:** a nontrivial fixed point is given by

$$\begin{aligned} r^* &= -\frac{\epsilon}{6}, \\ \lambda^* &= \frac{16\pi^2}{3}\epsilon. \end{aligned}$$

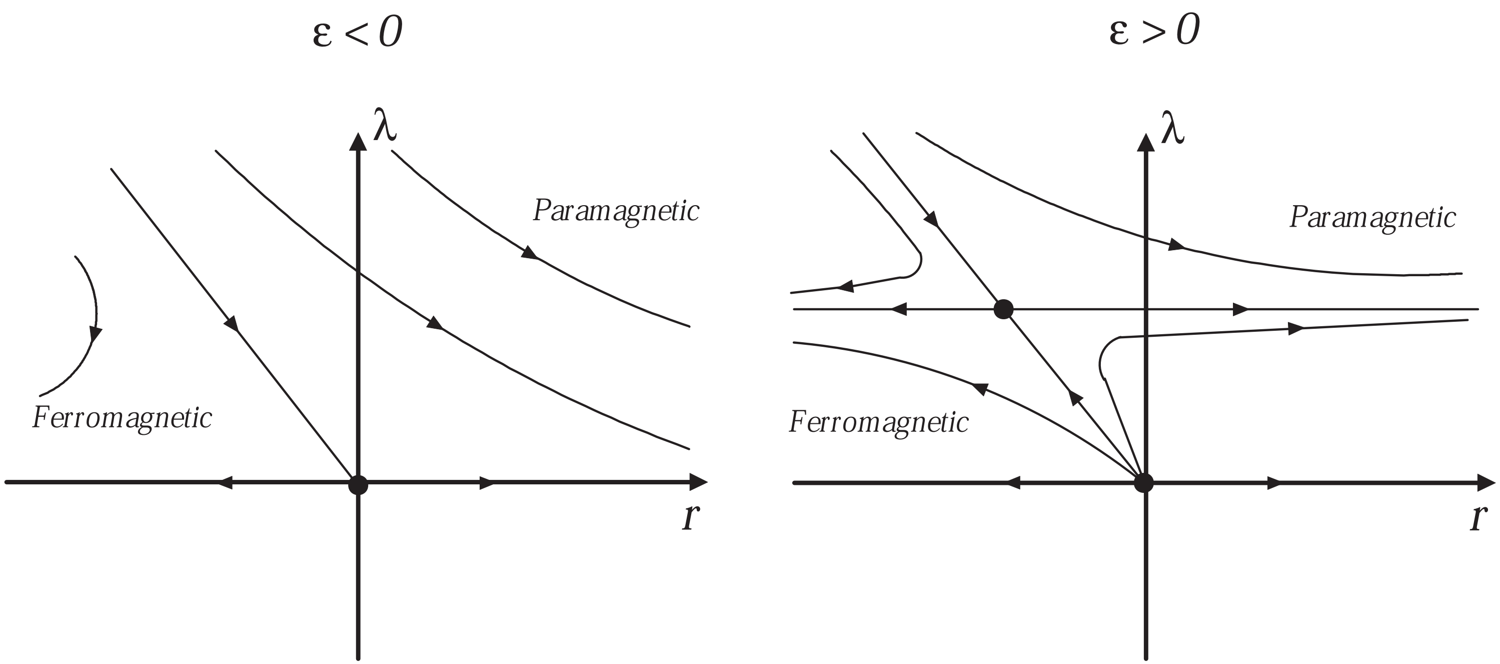
Around this fixed point, we have

$$
\begin{aligned}
y_t &= 2 - \frac{\epsilon}{3}, \\
y_h &= \frac{6 - \epsilon}{2}.
\end{aligned}
$$

The critical exponents can be computed as follows,

$$
\begin{aligned}
\alpha &= \frac{\epsilon}{3}, \\
\beta &= \frac{1}{2} - \frac{\epsilon}{6}, \\
\gamma &= 1 + \frac{\epsilon}{6}, \\
\delta &= 3 + \epsilon, \\
\eta &= 0, \\
\nu &= \frac{1}{2} + \frac{\epsilon}{2}.
\end{aligned}
$$

- Phase diagram ($\epsilon = 4 - D$)



Example 3: Spinless fermions in one dimension

Spinless fermions on a one-dimensional lattice

Let us consider the following generic Hamiltonian for a spinless fermion system on a one-dimensional lattice, which can be written in terms of a non-interacting part and a four-fermion interacting part,

$$H = H_0 + H_I,$$

with

$$\begin{aligned} H_0 &= \int_{-\pi}^{\pi} \frac{dK}{2\pi} E(K) \psi^\dagger(K) \psi(K), \\ H_I &= \frac{1}{2!2!} \left[\prod_{i=1}^{4} \int_{-\pi}^{\pi} \frac{dK_i}{2\pi} \right] u(K_4, K_3, K_2, K_1) \psi^\dagger(K_4) \psi^\dagger(K_3) \psi(K_2) \psi(K_1), \end{aligned}$$

where K_i are lattice momenta, $E(K)$ is the energy dispersion, $u(K_4, K_3, K_2, K_1)$ can be abbreviated as $u(4321)$ and obeys the symmetry condition

$$u(4321) = -u(3421) = -u(4312).$$

In the case of $K_F \simeq \pi/2$, the dispersion $E(K)$ can be linearized at the low energy domain as

$$E(k) - E(K_F) = v_F k,$$

with

$$k = |K| - K_F.$$

Note that there are two low energy branches near $K = \pm K_F$ with left-going and right-going fermions, we can rewrite H_0 with a cutoff Λ,

$$H_0 = v_F \sum_{p=L,R} \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} \psi_p^\dagger(k) k \psi_p(k), \tag{1}$$

where L and R denote left and right branch respectively.

Correspondingly, H_I can be further written as

$$H_I = \frac{1}{2!2!} \sum_{p_1 p_2 p_3 p_4 = L,R} \int_K^{\Lambda} u_{p_4 p_3 p_2 p_1}(4,3,2,1) \psi_{p_4}^{\dagger}(4) \psi_{p_3}^{\dagger}(3) \psi_{p_2}(2) \psi_{p_1}(1), \quad (2)$$

where

$$\int_K^{\Lambda} \equiv \prod_{i=1}^{4} \int_{-\Lambda}^{\Lambda} \frac{dk_i}{2\pi} \delta_{2\pi}(\epsilon_{p_1}(K_F + k_1) + \epsilon_{p_2}(K_F + k_2) - \epsilon_{p_3}(K_F + k_3) - \epsilon_{p_4}(K_F + k_4)),$$

and

$$\epsilon_p = \pm 1 \text{ for } R, L,$$

and $\delta_{2\pi}$ is the period 2π delta function.

Example 3: Spinless fermions in one dimension

Gaussian fixed point

Firstly, we only consider the non-interacting part. The action is

$$S_0 = \sum_{p=L,R} \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \bar{\psi}_p(k,\omega)(i\omega - v_F k)\psi_p(k,\omega), \tag{3}$$

where ω is the Matsubara frequency and we have taken the zero temperature limit $\beta \to \infty$. We then integrate out all $\psi(k,\omega)$ and $\bar{\psi}(k,\omega)$ in the momentum shell,

$$\Lambda/b \le |k| \le \Lambda,$$

and all ω. To make this action a fixed point, we define rescaled variables,

$$\begin{aligned} k' &= bk, \\ \omega' &= b\omega, \\ \psi'_p(k',\omega') &= b^{-3/2}\psi_p(k,\omega). \end{aligned}$$

Ignoring a constant that comes from rewriting the measure in terms of the new fields, we see that S_0 is invariant under the mode elimination and rescaling operations.

Example 3: Spinless fermions in one dimension

Quadratic perturbations

Now, we consider possible quadratic perturbations of the form,

$$\delta S_2 = \sum_{p=L,R} \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \mu(k,\omega) \bar{\psi}_p(k,\omega) \psi_p(k,\omega).$$

Rescaling momenta and fields, we find that

$$\mu'(k',\omega') = b\mu(k,\omega).$$

Let us expand μ in a Taylor series,

$$\mu(k,\omega) = \mu_{00} + \mu_{10} k + \mu_{01} i\omega + \cdots + \mu_{nm} k^n (i\omega)^m + \cdots .$$

$$\mu(k,\omega) = \mu_{00} + \mu_{10}k + \mu_{01}i\omega + \cdots + \mu_{nm}k^n(i\omega)^m + \cdots .$$

Thus

$$\begin{aligned}
\mu_{00} &\to b\mu_{00}, \\
\mu_{10} &\to \mu_{10}, \\
\mu_{01} &\to \mu_{01}, \\
&\quad \cdots , \\
\mu_{nm} &\to b^{1-n-m}\mu_{nm}, \\
&\quad \cdots
\end{aligned}$$

The only relevant term is μ_{00}, which reflects the readjustment of the Fermi sea to a change in chemical potential.

Two marginal terms can be absorbed to the linear dispersion.

Higher order terms are irrelevant under RG.

Example 3: Spinless fermions in one dimension

Quartic perturbations at tree-level

Then we turn to the quadratic interaction,

$$\delta S_4 = \frac{1}{2!2!} \sum_{p_1 p_2 p_3 p_4 = L,R} \int_{K\omega}^{\Lambda} u_{p_4 p_3 p_2 p_1}(4,3,2,1) \bar{\psi}_{p_4}(4) \bar{\psi}_{p_3}(3) \psi_{p_2}(2) \psi_{p_1}(1), \tag{4}$$

where

$$\begin{aligned} \int_K^{\Lambda} \equiv & \prod_{i=1}^{4} \int_{-\Lambda}^{\Lambda} \frac{dk_i}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega_i}{2\pi} \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4) \\ & \times \delta_{2\pi}(\epsilon_{p_1}(K_F + k_1) + \epsilon_{p_2}(K_F + k_2) - \epsilon_{p_3}(K_F + k_3) - \epsilon_{p_4}(K_F + k_4)), \end{aligned}$$

and

$$\epsilon_p = \pm 1 \text{ for } R, L.$$

The proceed of RG transformation is very similar to the ϕ^4 theory. Integrating out the momentum shell $\Lambda/b \le |k| \le \Lambda$, one has the leading terms of the form

$$\langle \delta S_4 \rangle = \frac{1}{2!2!} \left\langle \int_{K\omega}^{\Lambda/b} u(4321) \left(\bar{\psi}_< + \bar{\psi}_>\right)_4 \left(\bar{\psi}_< + \bar{\psi}_>\right)_3 (\psi_< + \psi_>)_2 (\psi_< + \psi_>)_1 \right\rangle_{0>} .$$

We would like to consider the order-u terms at tree-level at first, say, the terms of $\bar{\psi}_<\bar{\psi}_<\psi_<\psi_<$, which can be written as

$$\begin{aligned}
\delta S'_{4,tree} &= \frac{1}{2!2!} \int_{K\omega}^{\Lambda/b} u(4321)\bar{\psi}_<(4)\bar{\psi}_<(3)\psi_<(2)\psi_<(1) \\
&= \frac{1}{2!2!} (b^{\frac{3}{2}})^4 (b^{-2})^{4-1} \int_{K\omega}^{\Lambda} u'(4321)\bar{\psi}'(4)\bar{\psi}'(3)\psi'(2)\psi'(1) \\
&= \frac{1}{2!2!} \int_{K\omega}^{\Lambda} u'(4321)\bar{\psi}'(4)\bar{\psi}'(3)\psi'(2)\psi'(1).
\end{aligned}$$

where the factor $(b^{\frac{3}{2}})^4$ comes from $\bar{\psi}_<\bar{\psi}_<\psi_<\psi_<$ and $(b^{-2})^{4-1}$ comes from $\int_{K\omega}^{\Lambda/b}$.

Note that all K_F's cancel in non-Umklapp processes or get swallowed up in multiplies of 2π in Umklapp scattering processes. As a result, the momentum delta function $\delta_{2\pi}$ is free of K_F and scale well under the RG transformation. (Situation will change in $D > 1$.)

Therefore, we have

$$u'_{p_4p_3p_2p_1}(k'_i,\omega'_i) = u_{p_4p_3p_2p_1}(k_i,\omega_i).$$

Expanding $u(k_i,\omega_i)$ as

$$u(k_i,\omega_i) = u_0 + u_1(k_i,\omega_i) + u_2(k_i,\omega_i)^2 + \cdots ,$$

we find that the constant term u_0 is marginal and higher order terms are all irrelevant at tree-level. Thus, u depends on its descrete labels,

$$u_0 = u_{LRLR} = u_{RLRL} = -u_{LRRL} = -u_{RLLR}.$$

Other terms like u_{LLRR} are wiped out by the Pauli exclusive principle. The above analysis can be extended to six-fermion vertex or more. All these are irrelevant at tree-level.

- **Nearest-neighboring interaction:** As a concrete example, we consider that the u comes from the nearest-neighboring interaction,

$$\frac{u(4,3,2,1)}{2!2!} = U_0 \sin\frac{K_1 - K_2}{2} \sin\frac{K_3 - K_4}{2} \cos\frac{K_1 + K_2 - K_3 - K_4}{2}.$$

  - If K_1 and K_2 come from the same branch, e.g., R, we find that

$$\sin\frac{K_1 - K_2}{2} = \sin\frac{k_1 - k_2}{2} \simeq \frac{k_1 - k_2}{2}.$$

    It will result in an irrelevant coupling.

  - If K_1 and K_2 come from opposite branches,

$$\sin\frac{K_1 - K_2}{2} = \sin\left[\frac{\pi}{2} + O(k)\right].$$

    It will lead to a marginal interaction u_0 indepent of k.

Example 3: Spinless fermions in one dimension

One-loop RG: The Luttinger liquid

Let us examine the ultimate fate of the coupling constant u_0, marginal at tree-level, through one-loop RG.

- **Subdivision of fields**

By separating $\psi = \psi_< + \psi_>$, up to the one-loop (first order in $\bar{\psi}_> \psi_>$), we have

$$S[\bar{\psi}, \psi] = S_<[\bar{\psi}_<, \psi_<] + S_>[\bar{\psi}_>, \psi_>] + S_I[\bar{\psi}_<, \psi_<, \bar{\psi}_>, \psi_>],$$

with

$$
\begin{aligned}
S_<[\bar\psi_<,\psi_<] \;&=\; \sum_{p=L,R}\int_{-\Lambda/b}^{\Lambda/b}\frac{dk}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\bar\psi_p(k,\omega)(i\omega-v_Fk+\mu)\psi_p(k,\omega)\\
&\quad+\frac{1}{2!2!}\sum_{p_1p_2p_3p_4=L,R}\int_{K\omega}^{\Lambda/b}u_{p_4p_3p_2p_1}\bar\psi_{p_4}(4)\bar\psi_{p_3}(3)\psi_{p_2}(2)\psi_{p_1}(1),\\
S_>[\bar\psi_>,\psi_>] \;&=\; \sum_{p=L,R}\int_{\Lambda/b\le|k|\le\Lambda}\frac{dk}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\bar\psi_p(k,\omega)(i\omega-v_Fk+\mu)\psi_p(k,\omega),
\end{aligned}
$$

and

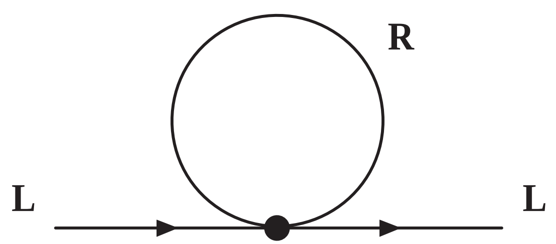
$$
\begin{aligned}
S_I[\bar\psi_<,\psi_<,\bar\psi_>,\psi_>] \;&=\; \frac{1}{2!2!}\sum_{p_1p_2p_3p_4=L,R}\int_{K\omega}^{\Lambda}u_{p_4p_3p_2p_1}\Big[\bar\psi_>(4)\bar\psi_>(3)\psi_<(2)\psi_<(1)\\
&\quad+\bar\psi_<(4)\bar\psi_<(3)\psi_>(2)\psi_>(1)+\bar\psi_>(4)\bar\psi_<(3)\psi_>(2)\psi_<(1)\\
&\quad+\bar\psi_<(4)\bar\psi_>(3)\psi_<(2)\psi_>(1)+\bar\psi_>(4)\bar\psi_<(3)\psi_<(2)\psi_>(1)\\
&\quad+\bar\psi_<(4)\bar\psi_>(3)\psi_>(2)\psi_<(1)\Big],
\end{aligned}
$$

where we have put the relevant term $\mu\bar\psi\psi$ in the action.

- **One-loop RG: chemical potential μ**

The one-loop contribution to the $\mu\bar{\psi}\psi$ term is given by the tadepole graph, resulting in

$$\begin{aligned}\mu' &= b\left[\mu - u_0 \int_{\Lambda/b\le|k|\le\Lambda} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{i\omega - v_F k}\right] \\ &= b\left[\mu - u_0 \int_{\Lambda/b\le|k|\le\Lambda} \frac{dk}{2\pi}\theta(-k)\right] \\ &= b\left[\mu - u_0 \frac{\Lambda}{2\pi}(1 - b^{-1})\right].\end{aligned}$$



Thus, the **flow equation** for μ becomes

$$\frac{d\mu}{dl} = \mu - \frac{\Lambda u_0}{2\pi}. \qquad (5)$$

The **fixed point** should be given by

$$\mu^* = \frac{\Lambda u_0^*}{2\pi}.$$

So that we can replace the original action by

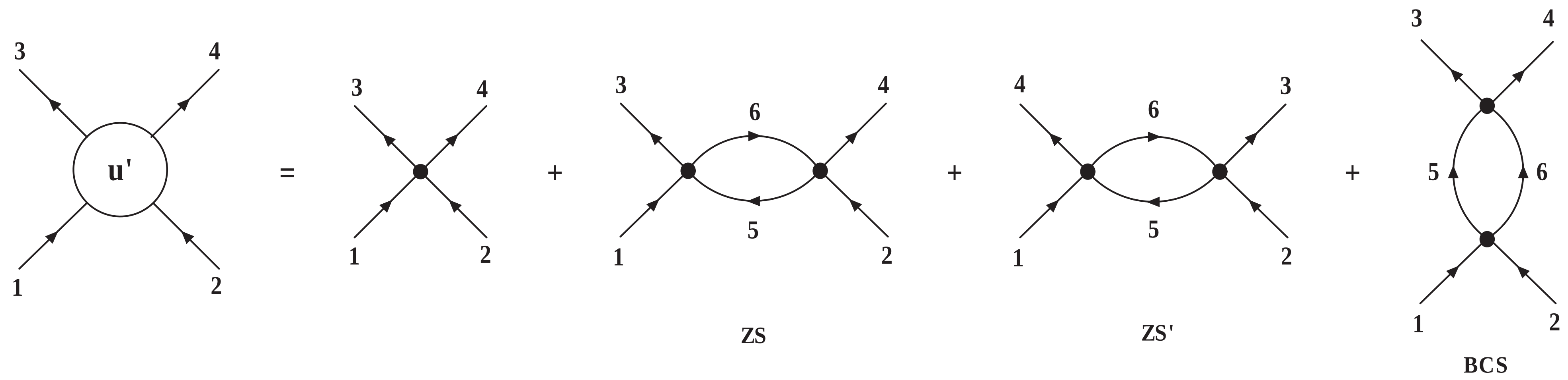
$$S = \bar{\psi}(i\omega - k)\psi + \frac{\Lambda u_0}{2\pi}\bar{\psi}\psi + \frac{u_0}{2!2!}\bar{\psi}\bar{\psi}\psi\psi$$

schematically. An RG transformation on this action will not generate the tradpole graph contribution.

- **A very important point:** We must fine-tune the chemical potential as a function of u to retain the same particle density. To be precise, we are fixing K_F. If we kept μ at the old value of zero, the system would flow away from the fixed point with K_F, not to a state with a gap, but to another gapless one with a smaller value of K_F. It means that the particle number changes.

- **One-loop RG: vertex u_0**

Remark: Functional RG will treat the vertex u_0 as a function $u(4321)$



Then we study **the one-loop contribution to the vertex u_0.** There are three types of Feynman diagrams, called ZS (zero sound), ZS', and BCS graph.

Formally, we have

$$\begin{aligned} du(4321) &= \int u(6351)u(4526)G(5)G(6)\delta(3+6-1-5)d5d6 \\ &\quad - \int u(6451)u(3526)G(5)G(6)\delta(4+6-1-5)d5d6 \\ &\quad - \frac{1}{2}\int u(6521)u(4365)G(5)G(6)\delta(5+6-1-2)d5d6 \qquad (6) \end{aligned}$$

where

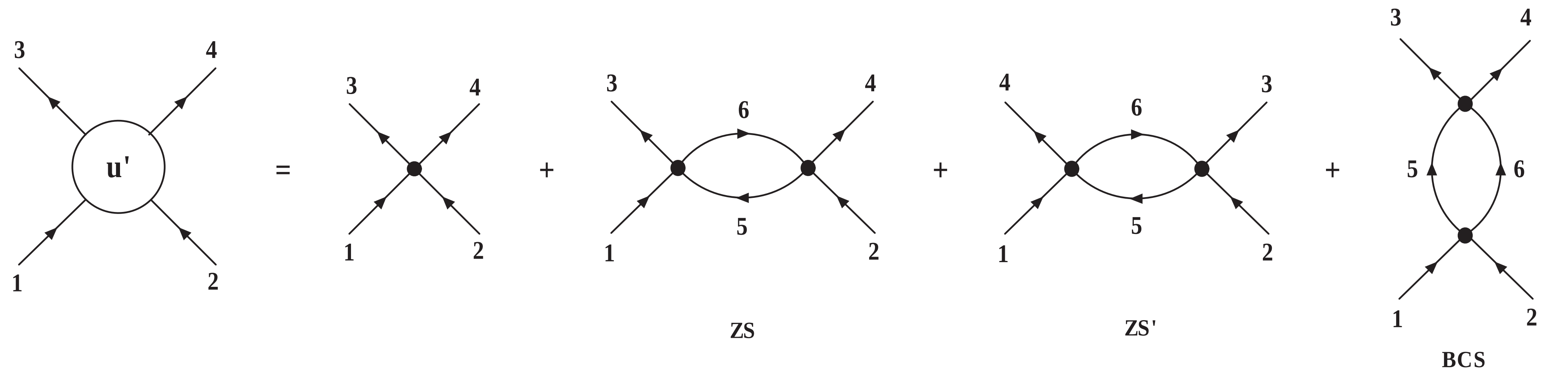
$$G^{-1} = i\omega - k - \frac{\Lambda u_0}{2\pi}.$$

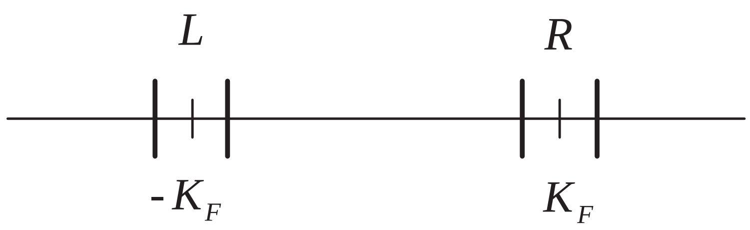
As discussed in previous slides, u_{LLRR} like term has been wiped by the Pauli exclusive principle, we only consider terms with opposite 1 and 2, and opposite 3 and 4. For example, $(4321) = (LRLR)$. We can set all external lines with $k = 0$ and $\omega = 0$, since the marginal coupling u does not depend on momentum and frequency.

So that

(1) the loop frequencies in the ZS and ZS' graph are equal, while those in the BCS graph are equal and opposite.

(2) The momentum transfers at the left vertex are $Q = K_1 - K_3 = 0$ in the ZS graph and $Q' = K_1 - K_4 = \pm 2K_F$ in the ZS' graph.





Therefore if the one loop momentum $5 = K$ lies in any of the four shell in the figure, so does the other loop momentum 6, which equals to K, $K + 2K_F$ or $-K$ in the ZS, ZS' and BCS graphs respectively.
Then we can eliminate the momentum-conserving delta function using $\int d6$, and replace 5 by K, resulting in

$$
\begin{aligned}
du(LRLR) &= \int_{-\infty}^{\infty} \int_{d\Lambda} \frac{d\omega dK}{4\pi^2} \frac{u(KRKR)u(LKLK)}{[i\omega - E(K)][i\omega - E(K)]} \\
&\quad - \int_{-\infty}^{\infty} \int_{d\Lambda} \frac{d\omega dK}{4\pi^2} \frac{u(K'LKR)u(RKLK')}{[i\omega - E(K)][i\omega - E(K')]} \\
&\quad - \frac{1}{2} \int_{-\infty}^{\infty} \int_{d\Lambda} \frac{d\omega dK}{4\pi^2} \frac{u(-KKLR)u(LR-KK)}{[i\omega - E(K)][-i\omega - E(-K)]} \\
&\equiv ZS + ZS' + BCS.
\end{aligned} \tag{7}
$$

where $K' = K \pm 2K_F$.

With the help of

$$\begin{aligned} E(-K) &= E(K), \\ E(K' = K \pm 2K_F) &= -E(K), \end{aligned}$$

we have

$$\begin{aligned} ZS' &= -\int_{-\infty}^{\infty}\int_{d\Lambda\in L}\frac{d\omega dK}{4\pi^2}\frac{u(K'LKR)u(RKLK')}{[i\omega - E(K)][i\omega + E(K)]}, \\ BCS &= \frac{1}{2}\int_{-\infty}^{\infty}\int_{d\Lambda}\frac{d\omega dK}{4\pi^2}\frac{u(-KKLR)u(LR-KK)}{[i\omega - E(K)][-i\omega - E(K)]}. \end{aligned}$$

It is noted that only $K \in L$ and $K' \in R$ gives rise to nonvanishing $u(K'LKR)u(RKLK')$ in the ZS' graph. Since

$$u(KRKR)u(LKLK) = 0,$$

we find that

$$ZS = 0.$$

We also have

$$\begin{aligned} ZS' &= -u^2 \int_{-\infty}^{\infty} \int_{d\Lambda \in L} \frac{d\omega dK}{4\pi^2} \frac{1}{[i\omega - E(K)][i\omega + E(K)]} \\ &= u^2 \int_{d\Lambda \in L} \frac{dK}{2\pi} \frac{1}{2|E(K)|} = -\frac{u^2}{2\pi v_F} \frac{d\Lambda}{\Lambda}, \end{aligned}$$

and

$$\begin{aligned} BCS &= -\frac{u^2}{2} \int_{-\infty}^{\infty} \int_{d\Lambda} \frac{d\omega dK}{4\pi^2} \frac{1}{[i\omega - E(K)][-i\omega - E(K)]} \\ &= -\frac{u^2}{2} \int_{d\Lambda} \frac{dK}{2\pi} \frac{1}{2|E(K)|} = \frac{u^2}{2\pi v_F} \frac{d\Lambda}{\Lambda}. \end{aligned}$$

So that

$$du(LRLR) = 0,$$

due to the cancellation of ZS' and BCS graph.

Therefore, u **is still marginal.**

- **Flow equations and fixed points**

The flow equations to one loop for μ and u are

$$\begin{aligned} \frac{d\mu}{dl} &= \mu - \frac{\Lambda u}{2\pi}, \\ \frac{du}{dl} &= 0. \end{aligned}$$

There is a line of fixed points,

$$\begin{aligned} \mu^* &= \frac{\Lambda u^*}{2\pi}, \\ u^* &= \text{arbitary.} \end{aligned}$$

Example 3: Spinless fermions in one dimension

- **Discussions:**

  – Usual mean-field theory, e.g. charge density wave (CDW) mean field theory only focus on the ZS' graph and ignore the BCS graph, resulting in a relevant u,

  $$\frac{du}{dl} = \frac{u^2}{2\pi v_F}.$$

  If the repulsive u grows, one expects a CDW instability.

  – Meanwhile, if one only considers the BCS graph,

  $$\frac{du}{dl} = -\frac{u^2}{2\pi v_F}.$$

  One expects a superconducting instability for the negative u.

  - If we were not at half filling, then the Umklapp process $(RR \leftrightarrow LL)$ is suppressed by a factor $(k_1 - k_2)(k_3 - k_4)$ at tree-leve and is irrelevant. The system will be at a scale-invariant state, called a **Luttinger liquid**. This liquid provides us with an example of where the RG does better than mean-field theory.

  - At half-filling, the Umklapp process $(RR \leftrightarrow LL)$ will become marginal and then relevant, leading to CDW or BCS instability.

- **Homework:** Derive equation (6).

References

- John Cardy, *Scaling and renormalization group in statistical physics*, Cambridge Univsersity Press (1996).
- R. Shankar, *Renormalization-group approach to interacting fermions*, Review of Modern Physics, 66, 129 (1994).
- A. M. Polyakov, *Gauge fields and Strings*, Harwood Academic Publishers, (1987).
- Alexander Altland and Ben Simons, *Condensed Matter Field theory*, Cambridge Univsersity Press (2010).
- Jean Zinn-Justin, *Phase Transition and Renormalization Group*, Oxford Univsersity Press (2002).
- M. E. Peskin and D. V. Schroeder, *An introduction to quantum field theory*, Westview Press (1995).