© 1975

Topological States of Matter

-- Eyeing on Quantum Information Science

Xin Wan
Zhejiang University

Organization

- Lecture 1 (8/28)
  - Topological phases of matter
  - Topological quantum computation
- Lecture 2 (8/29)
  - Matrix product states (MPS)
  - Infinite time-evolving block decimation (iTEBD)
  - MPS for FQH states

http://zimp.zju.edu.cn/~xinwan/Nanjing2016/

Outline

- Introduction
  - Quantum computation
  - Quantum error correction codes
  - Kitaev's toric codes – a topological phase
  - Topological quantum computation
- Examples of topological phases
- Implementation of topological quantum computation
- Summary

Classical vs Quantum Comp.

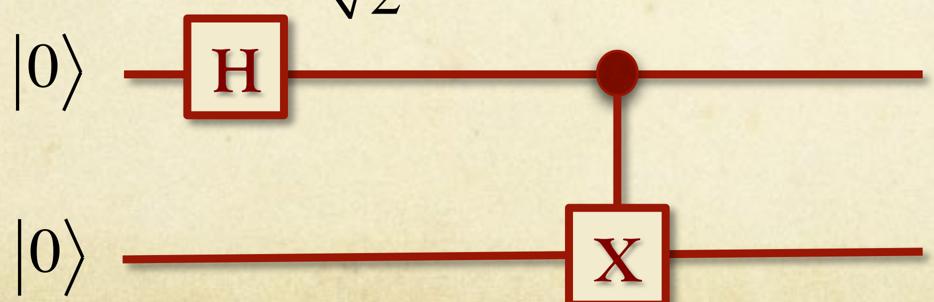
- Information encoded in **bits.**
- Possible bit states: 0 or 1

- Information encoded in **qubits.**
- Possible qubit states: any superposition described by wave function

$$|\Psi\rangle = a|0\rangle + b|1\rangle$$

superposition

$$\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$



$$\frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

entanglement

Classical vs Quantum Comp.

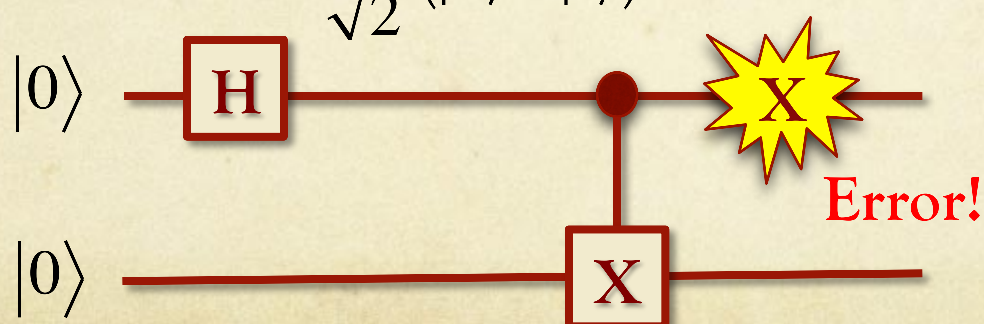
- Information encoded in **bits.**
- Possible bit states: 0 or 1

- Information encoded in **qubits.**
- Possible qubit states: any superposition described by wave function

$$|\Psi\rangle = a|0\rangle + b|1\rangle$$

superposition

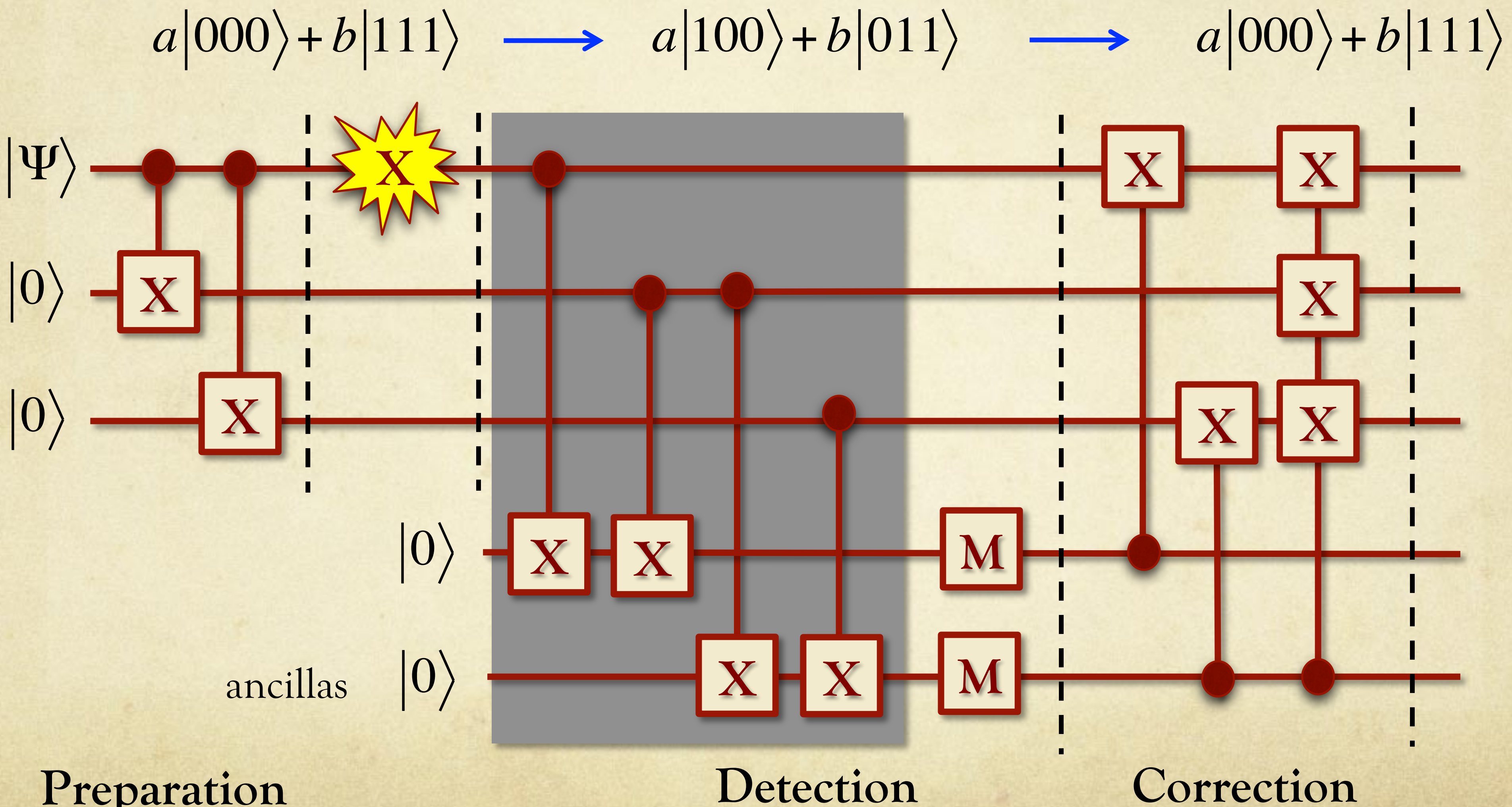
$$\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$



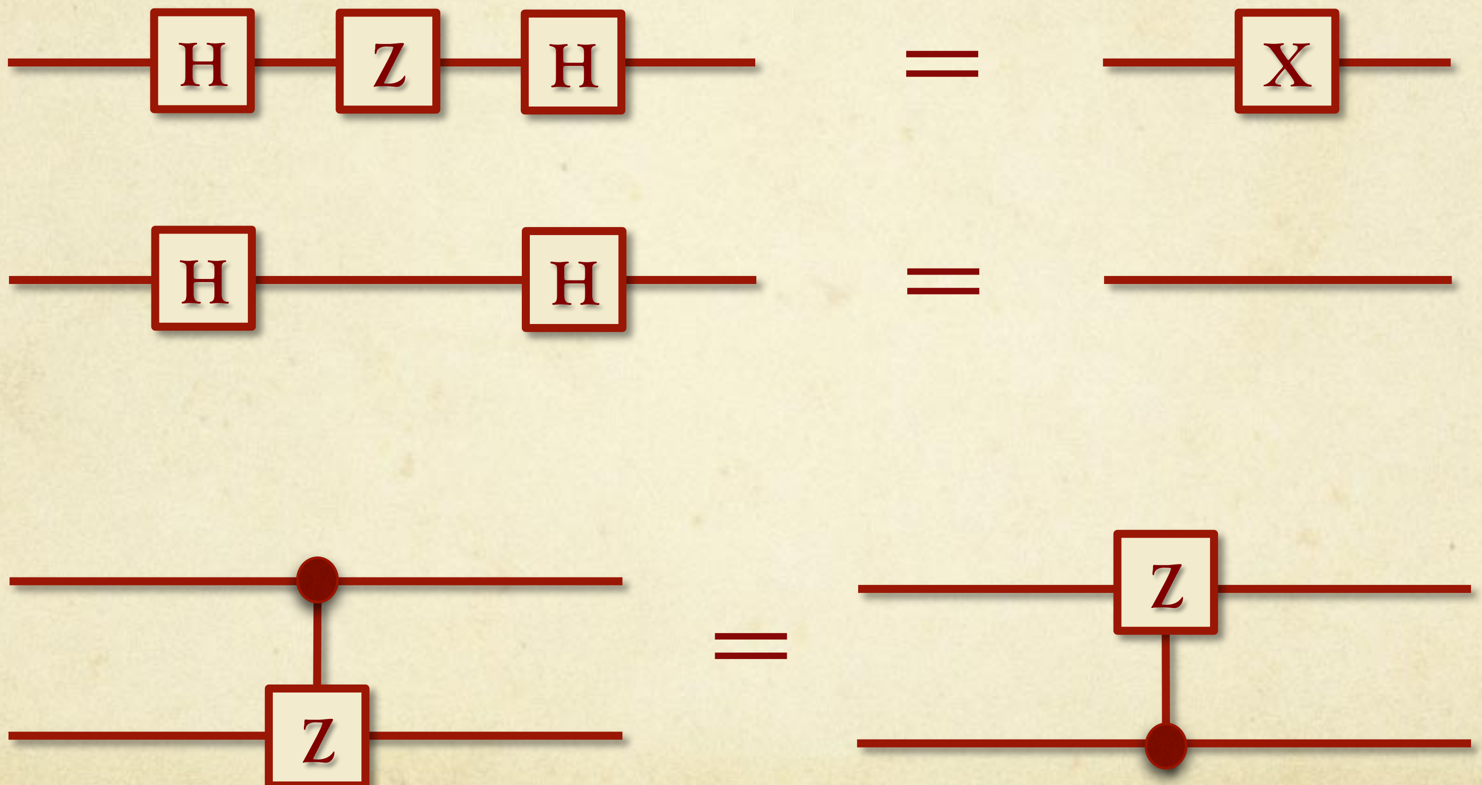
$$\frac{1}{\sqrt{2}}\left(|10\rangle + |01\rangle\right)$$

entanglement

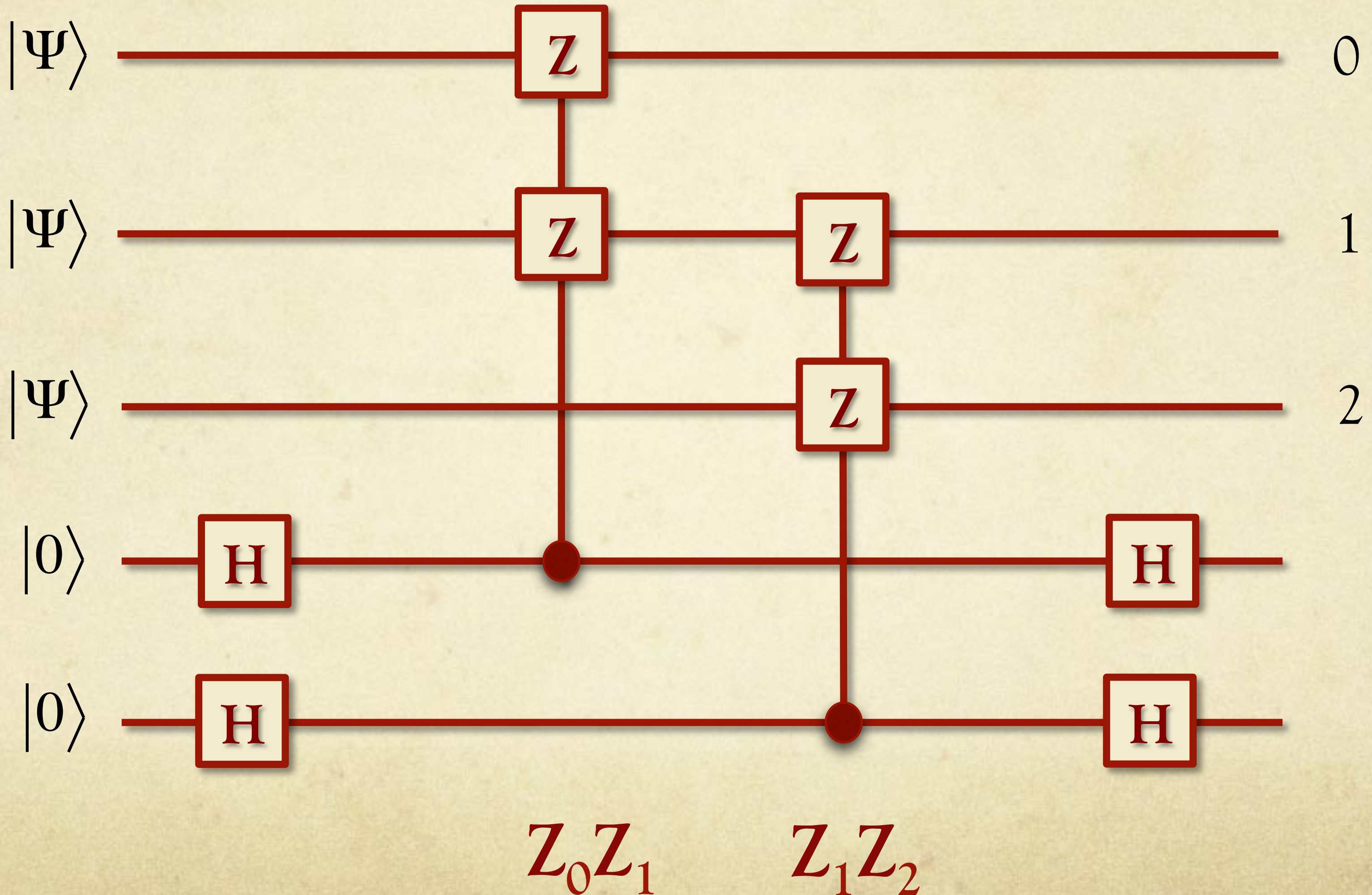
3-bit Error-Correction Code



Equalities



Circuit Equivalence



Error Detection

Type of Error (or No Error)	Z_0Z_1	Z_1Z_2
1	+	+
X_0	-	+
X_1	-	-
X_2	+	-

+: Commute -: Anticommute

Bases in a 2^3-dim space is separated into 4 sets. only 1 set is used to encode 1-bit of information. An X_i-error modifies the qubit into the non-encoding space

$$|\tilde{0}\rangle = |000\rangle$$

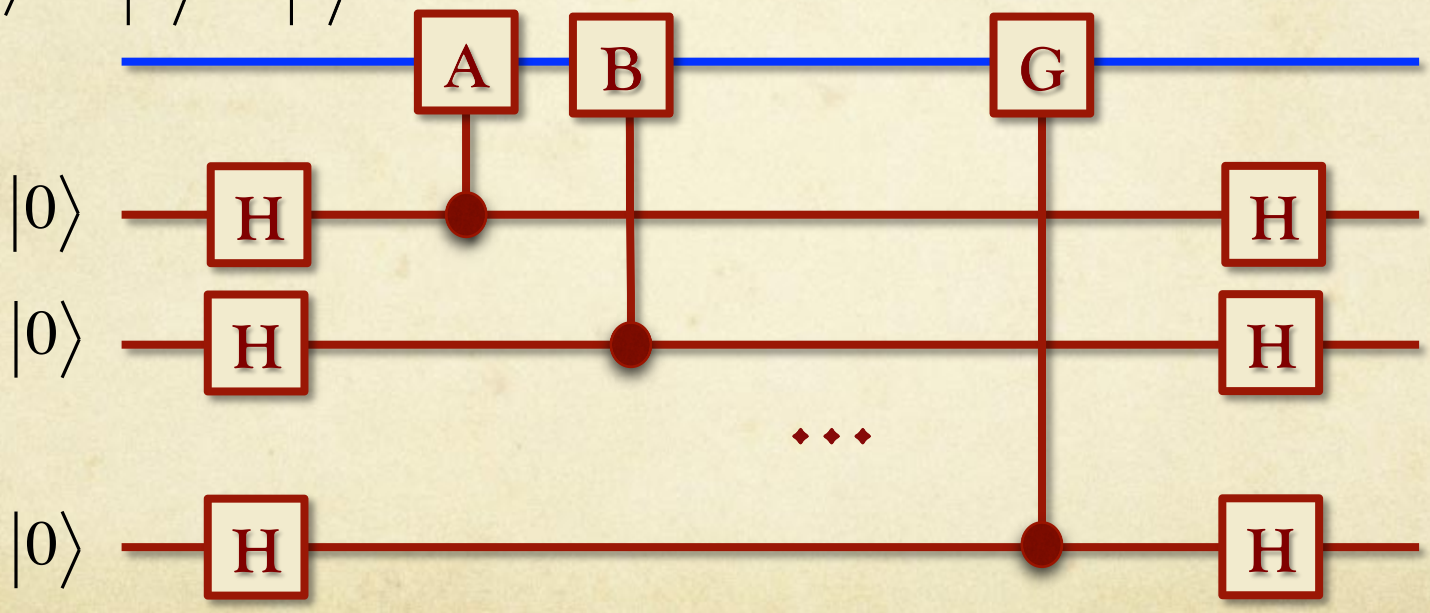
$$|\tilde{1}\rangle = |111\rangle$$

Stabilizers

- Error detection can be cast into an equivalent form with commuting operators A, B, C, ..., satisfying

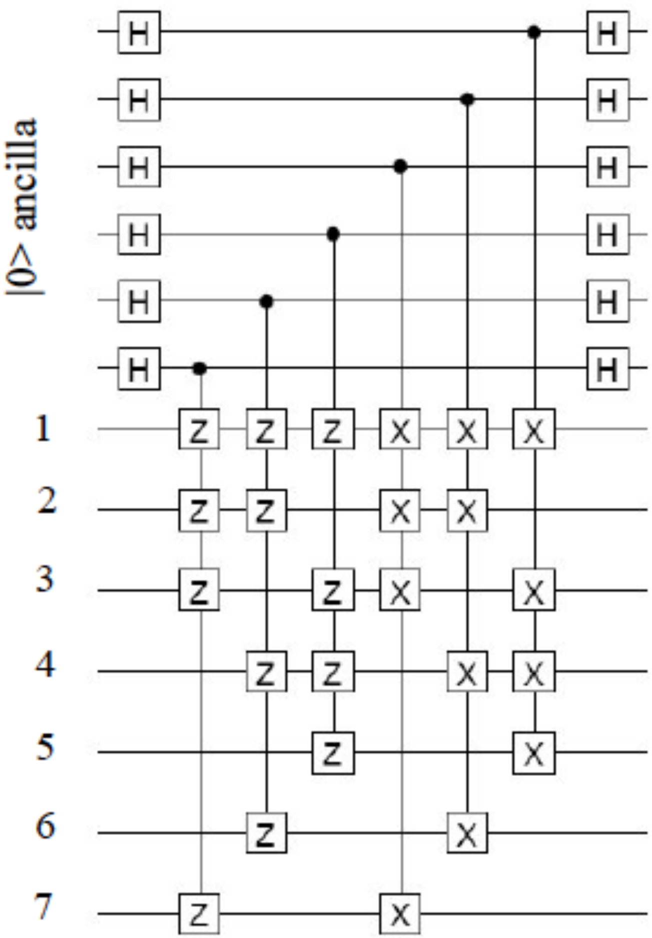
$$A^2 = B^2 = \cdots = G^2 = 1$$

$$|\Psi\rangle = a|\tilde{0}\rangle + b|\tilde{1}\rangle$$



$$|\tilde{0}\rangle = \frac{(1+A)}{\sqrt{2}}\frac{(1+B)}{\sqrt{2}}\cdots\frac{(1+G)}{\sqrt{2}}|00\cdots 0\rangle \qquad |\tilde{1}\rangle = \frac{(1+A)}{\sqrt{2}}\frac{(1+B)}{\sqrt{2}}\cdots\frac{(1+G)}{\sqrt{2}}|11\cdots 1\rangle$$

Steane's 7-qubit Code



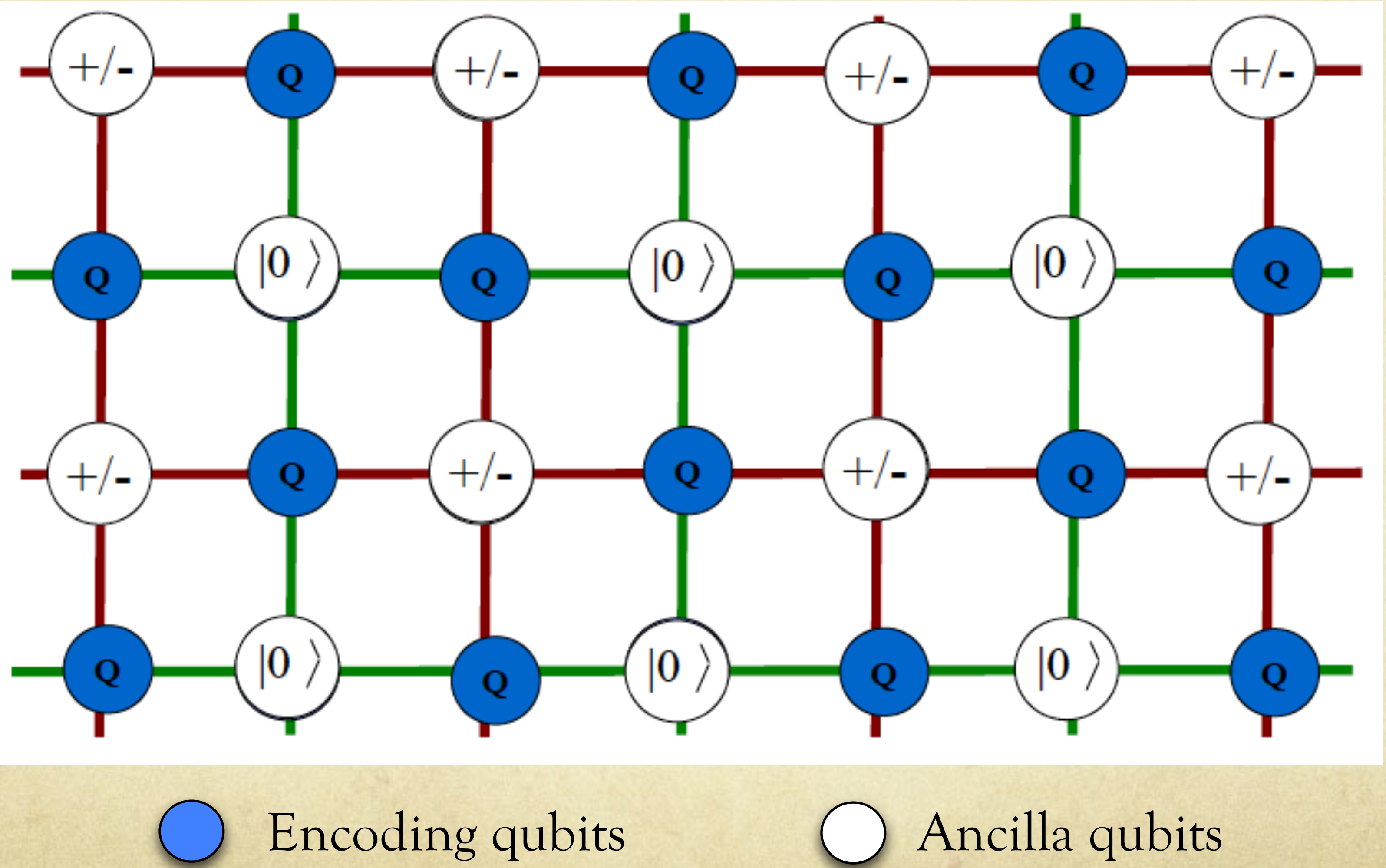
$$M(1) = Z_1 Z_2 Z_3 Z_7$$
$$M(2) = Z_1 Z_2 Z_4 Z_6$$
$$M(3) = Z_1 Z_3 Z_4 Z_5$$
$$M(4) = X_1 X_2 X_3 X_7$$
$$M(5) = X_1 X_2 X_4 X_6$$
$$M(6) = X_1 X_3 X_4 X_5$$

Error correction: Circuit does non-demolition measurement of operators

Disadvantages:

- Lots of qubits
- Long-distance couplings (regularity is not geometric)

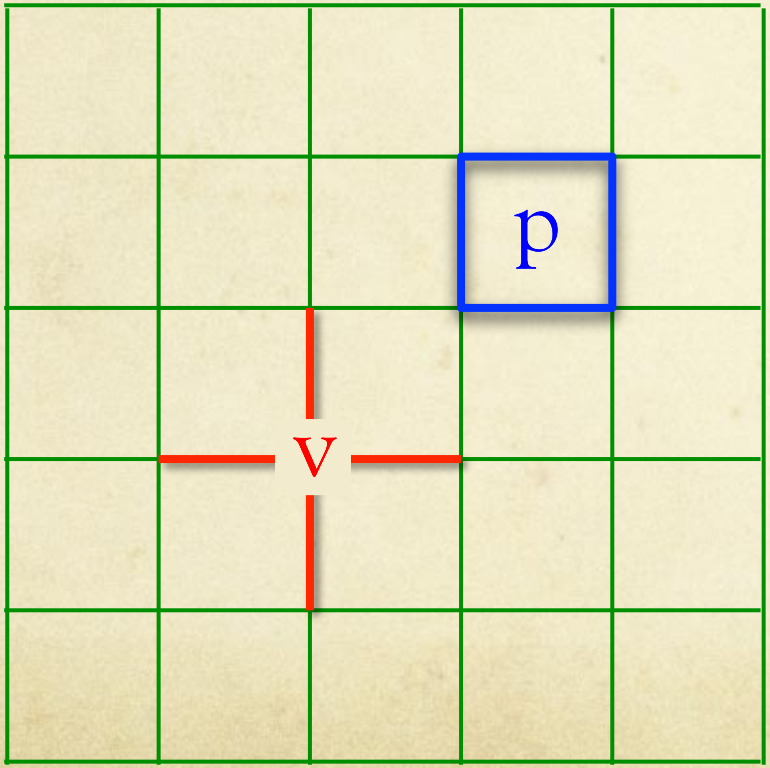
Surface Codes



See, e.g., Fowler et al., PRA 86, 032324 (2012)

Kitaev's Toric Codes

Square lattice, spins or qubits positioned at lattice edges
[Kitaev, Ann. Phys. 303, 2 (2003)]



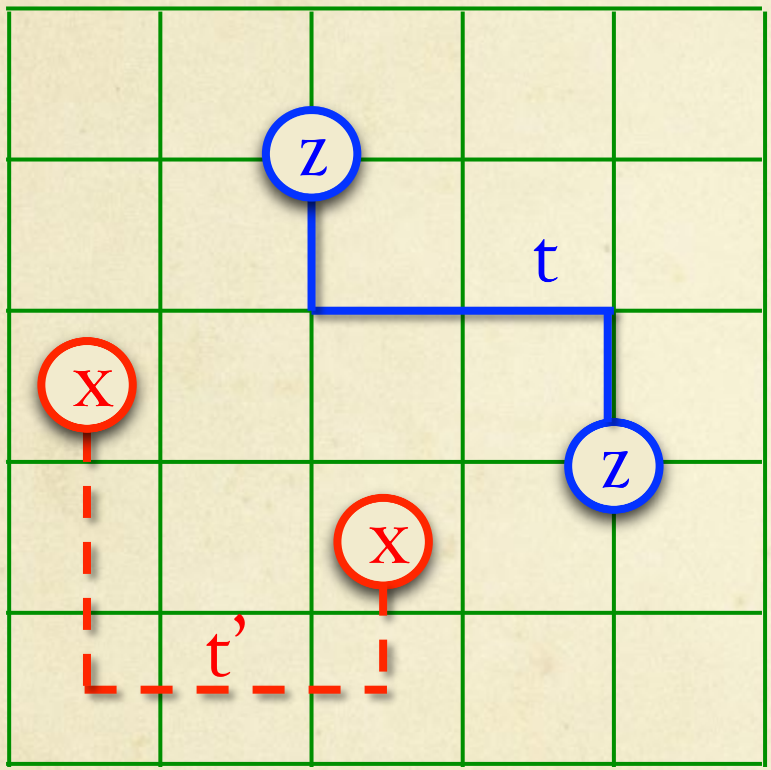
$$H_0 = -\sum_v A_v - \sum_p B_p$$

$$A_v = \prod_{j\in v} \sigma_j^x \qquad B_p = \prod_{j\in p} \sigma_j^z$$

Ground state: equal-weight superposition of all loop configurations

$$|\xi\rangle = \prod_v \frac{1}{\sqrt{2}}(1 + A_v)|00\cdots 0\rangle$$

Anyonic Excitations



$$\left|\psi^z(t)\right\rangle = S^z(t)\left|\xi\right\rangle$$

$$\left|\psi^x(t')\right\rangle = S^x(t')\left|\xi\right\rangle$$

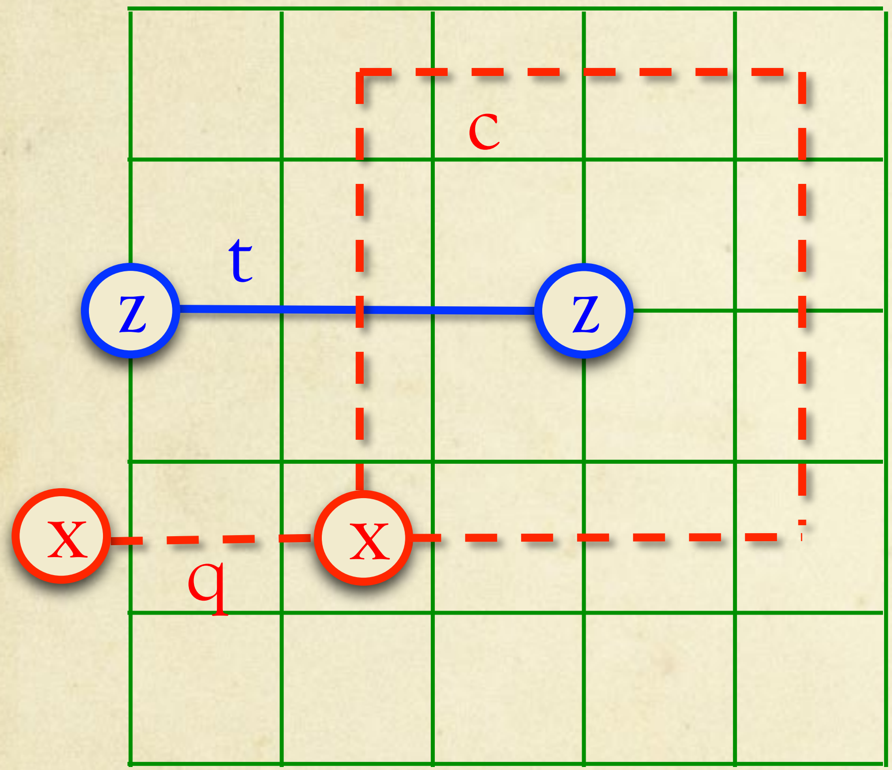
String operators:

$$S^z(t) = \prod_{j\in t} \sigma_j^z$$

$$S^x(t') = \prod_{j\in t'} \sigma_j^x$$

Square lattice, spins or qubits positioned at lattice edges
[Kitaev, Ann. Phys. 303, 2 (2003)]

Braiding



$$|\psi_{initial}\rangle = S^z(t)|\psi^x(q)\rangle$$

$$|\psi_{final}\rangle = S^x(c)S^z(t)|\psi^x(q)\rangle$$

$$= -|\psi_{initial}\rangle$$

because

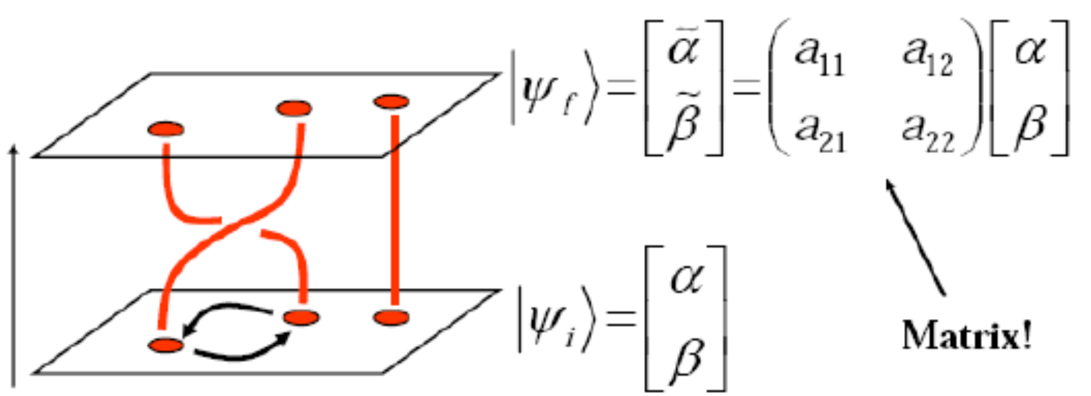
$$\{S^x(c), S^z(t)\} = 0$$

$$S^x(c)|\psi^x(q)\rangle = |\psi^x(q)\rangle$$

Square lattice, spins or qubits positioned at lattice edges
[Kitaev, Ann. Phys. 303, 2 (2003)]

Topological Quantum Computation

- Topological phases
  - Degenerate ground states protected by a spectral gap
  - Braiding of anyonic excitations = unitary evolution
  - Robust against noises (local perturbations)
  - Perform error correction on the physical level
- Topological quantum computation

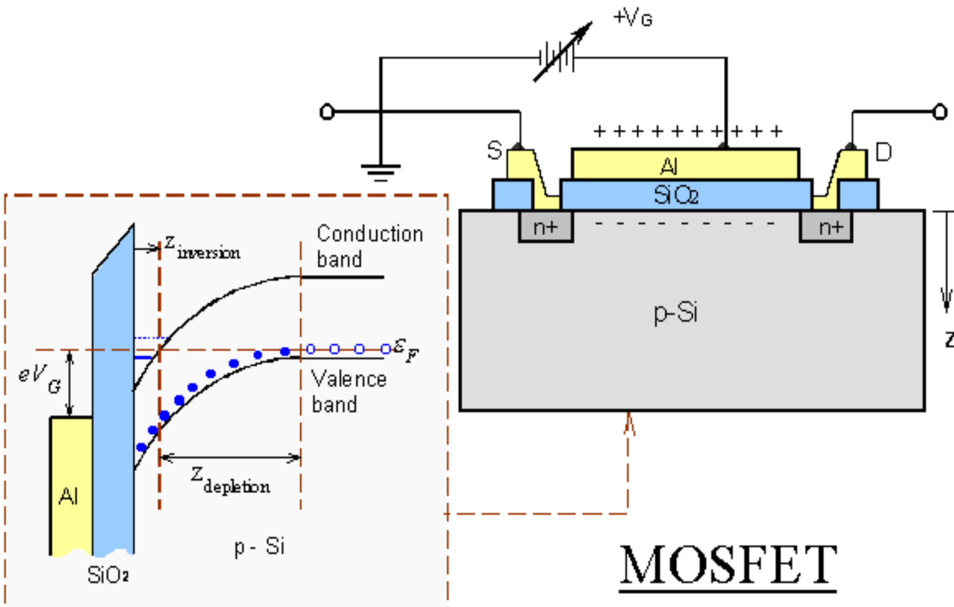


Matrices form a non-Abelian representation of the braid group.

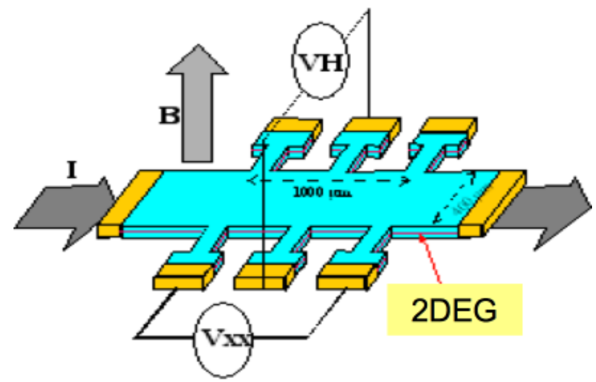
Outline

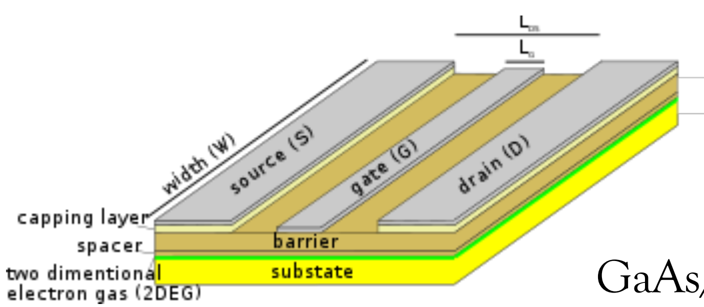
- Introduction
- Examples of topological phases
  - Integer quantum Hall states
  - Topological insulators & topological superconductors
  - Examples of interacting topological phases
- Implementation of topological quantum computation
- Summary

Two-Dimensional Electron Gases

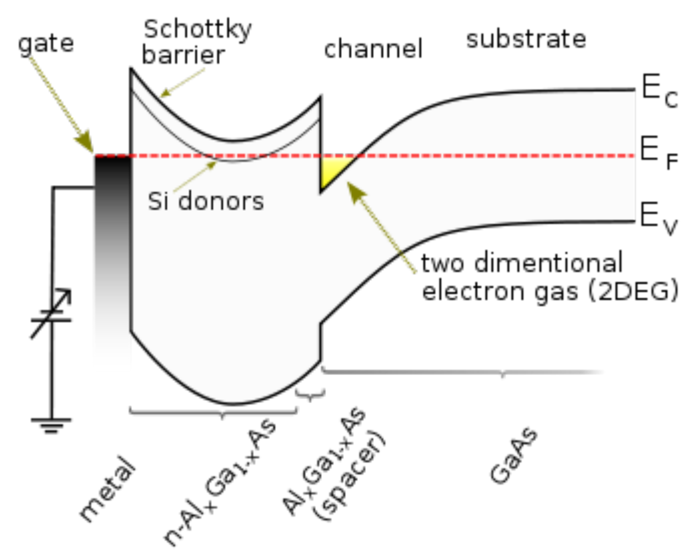


MOSFET





GaAs/AlGaAs heterostructure



Quantum Motion of Electrons in a B-Field

- 2D electrons in a perpendicular *B*: the quantization of the cyclotron motion → a discrete spectrum with macroscopically large degeneracy

Magnetic length:

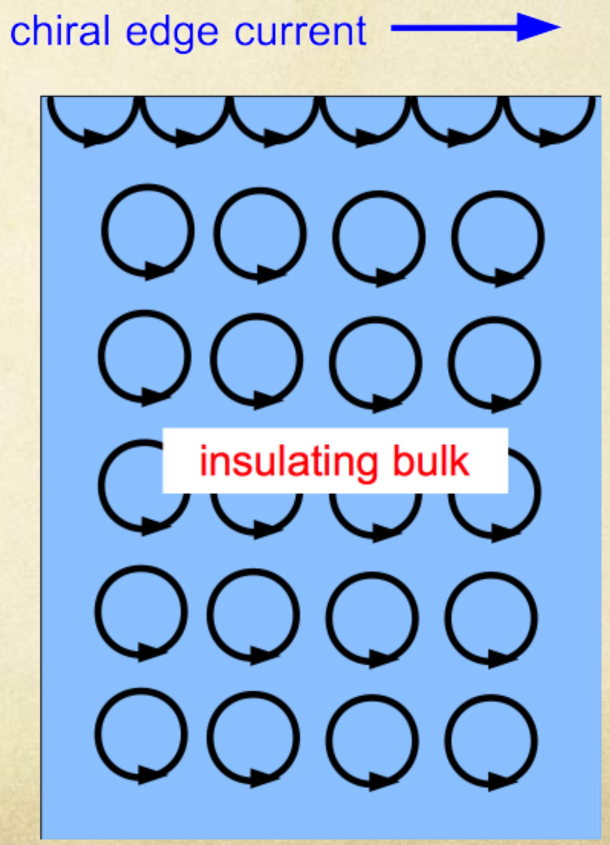
$$l_B = \sqrt{\frac{\hbar c}{eB}} \propto B^{-1/2}$$

Cyclotron frequency:

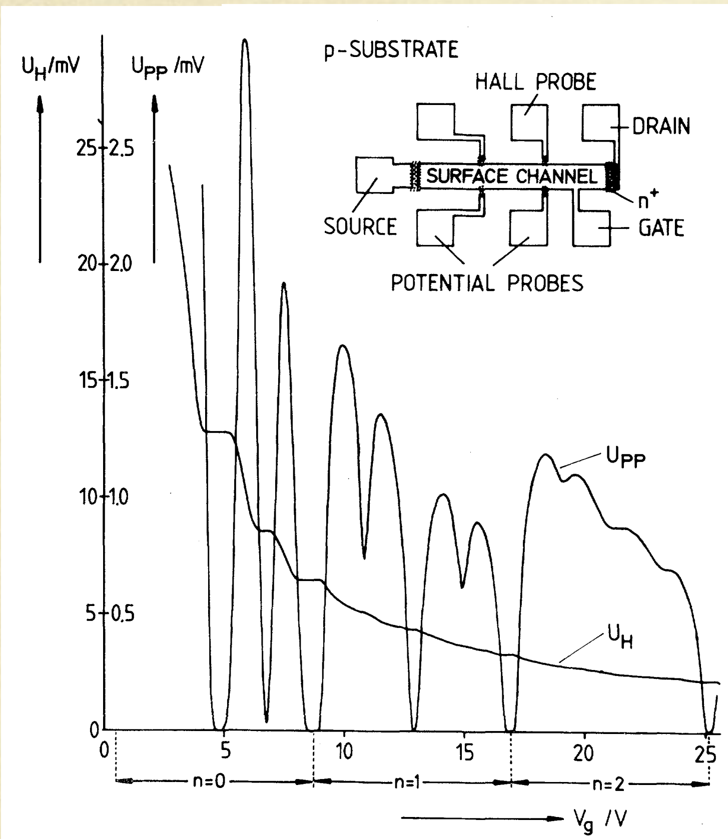
$$\omega_c = \frac{eB}{mc} \propto B$$

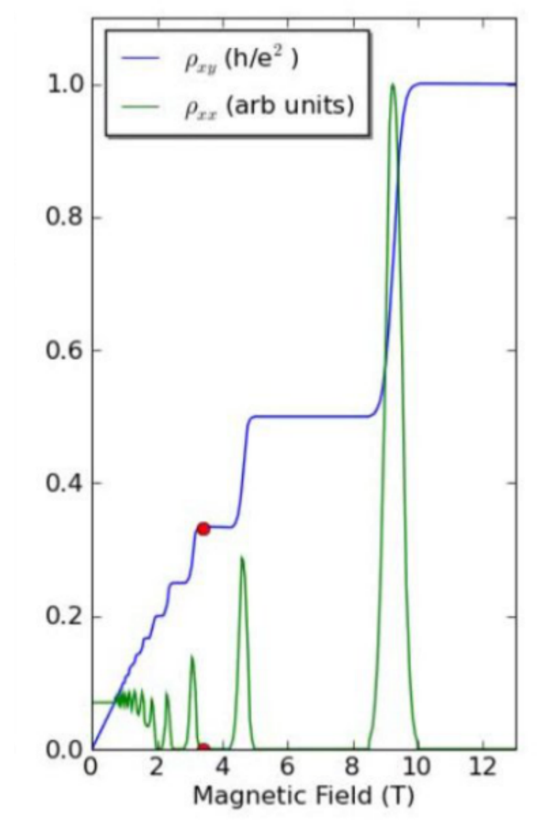
Landau level degeneracy:

$$N_\varphi = \frac{A}{2\pi l_B^2} = \frac{\Phi}{\Phi_0} \propto B$$



Integer Quantum Hall Effect (1980)





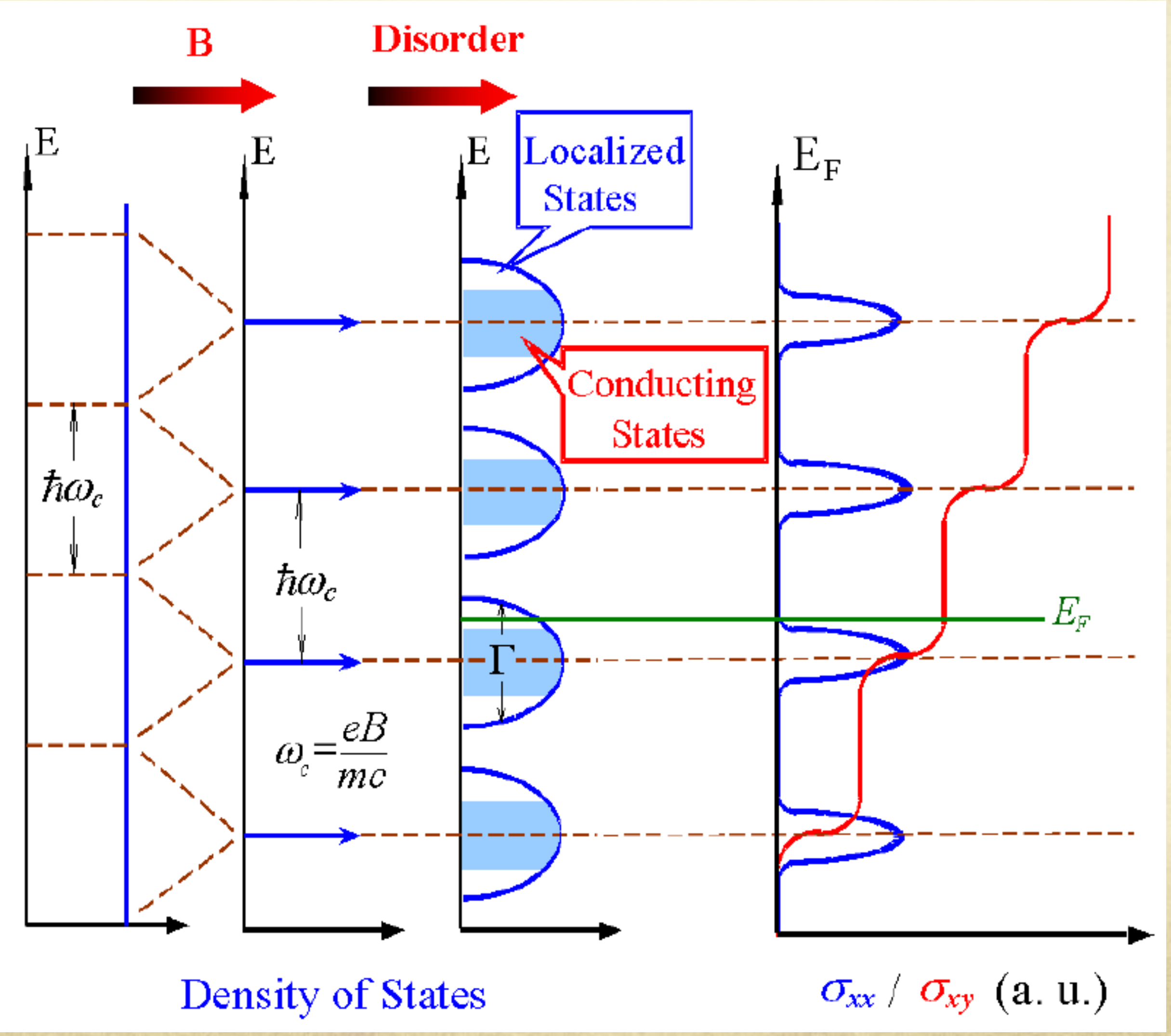
Klaus von Klitzing

$$\rho_{xy} = \frac{h}{n e^2}$$

$$\sigma_{xy} = \frac{n e^2}{h}$$

Nobel Prize 1985: "for the discovery of the quantized Hall effect"

Localization in Landau Levels



Difference in Quantum Numbers

- Quantum numbers related to symmetry
  - Example: angular momentum ← rotational symmetry
  - Degeneracy from the algebra of the group generators
  - Structure destroyed by symmetry breaking
- Quantum numbers determined by topology
  - Related to the winding number of a condensate wave function
  - Survive relatively strong perturbation

$$h/e^2 = 25{,}812.807449(86)\ \Omega$$

Hall Conductance as Curvature

- Hall conductance: Kubo formula derived from the linear-response theory

$$t(L_{x,y})|\psi\rangle = e^{i\theta_{x,y}}|\psi\rangle$$

$$\sigma_{xy}(m;\theta_x,\theta_y) = \frac{ie^2\hbar}{L_x L_y}\sum_{n\neq m}\frac{\langle\psi_m|v_x|\psi_n\rangle\langle\psi_n|v_y|\psi_m\rangle - h.c.}{(E_m-E_n)^2}$$

- Geometric interpretation of Hall conductance: local curvature in the boundary condition space or the flux space

$$\sigma_{xy}(m;\theta_x,\theta_y) \sim 2\Im\left\langle\frac{\partial\psi_m}{\partial\theta_x}\middle|\frac{\partial\psi_m}{\partial\theta_y}\right\rangle$$

Hall Conductance: A Topological Invariant

- Hall conductance averaged over a torus of boundary conditions is a quantized integral (Thouless *et al.*)

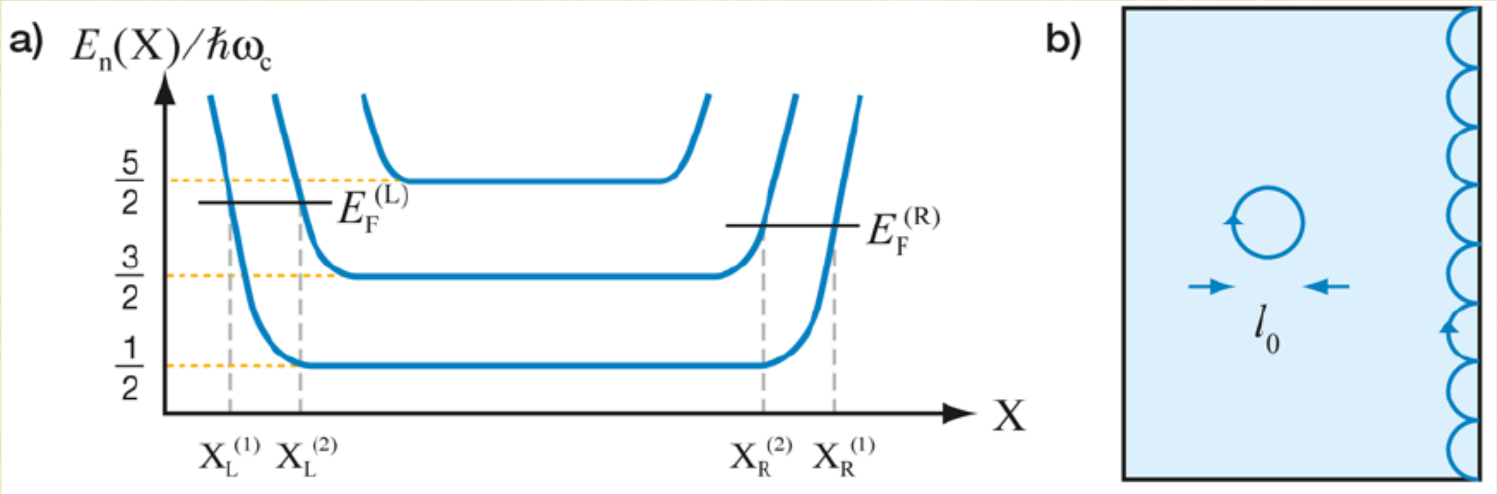
$$\sigma_{xy}(m) = \langle \sigma_{xy}(m;\theta_x,\theta_y)\rangle = \frac{e^2}{2\pi h}\int d\theta_x \int d\theta_y \, 2\Im \left\langle \frac{\partial \psi_m}{\partial \theta_x} \middle| \frac{\partial \psi_m}{\partial \theta_y} \right\rangle = C_1(m)\frac{e^2}{h}$$

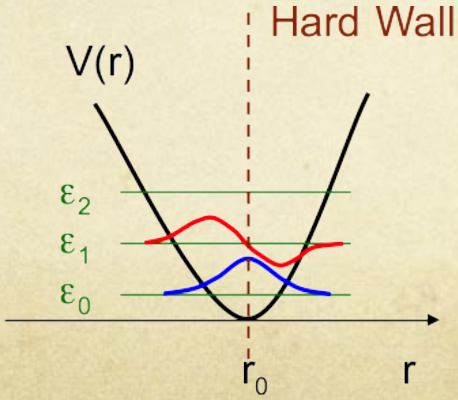
$$\frac{1}{2\pi}\int_S K\, dA = 2(1-g)$$

Gauss-Bonnet-Chern formula

- Topological: small perturbation → no change in Chern number
- Transitions between Chern numbers: level crossing (curvature diverges)

Edge States: Halperin (1982)



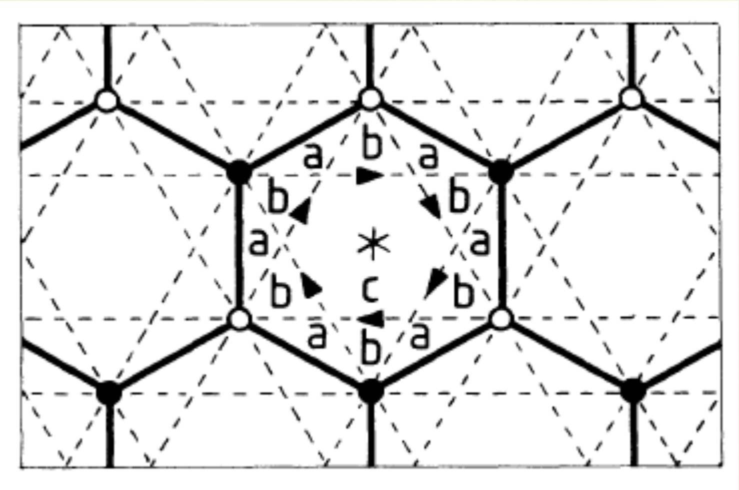


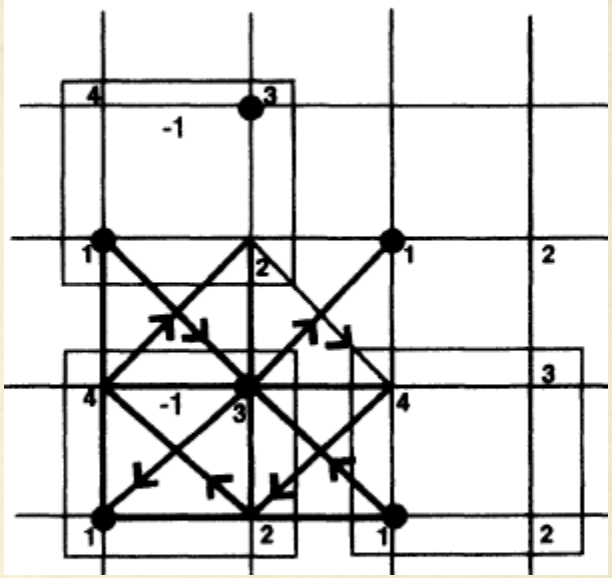
Chiral, gapless edge excitation

No backscattering !!
Chiral Fermi liquid

Quantum Anomalous Hall Effect

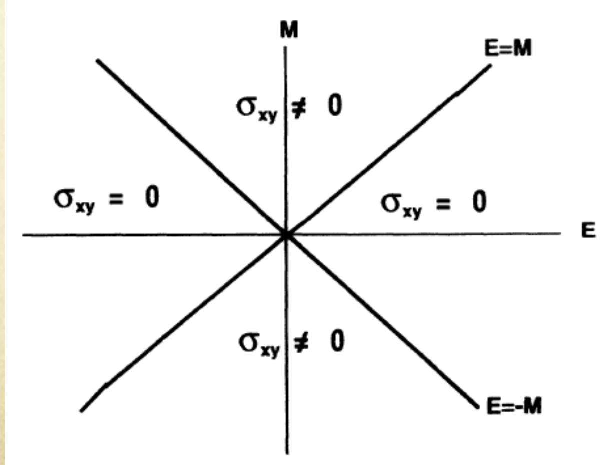
Haldane (1988); Ludwig et al. (1994)





Diagonal hopping:
T' -- breaks time-reversal symmetry

Staggered chemical potential:
m – breaks inversion symmetry



Dirac Model of IQHE

$$H = -i\partial_x \sigma_x - i\partial_y \sigma_y + m\sigma_z$$

Bloch wave function

$$\left|u^+(k)\right\rangle = \frac{1}{\sqrt{2\lambda(\lambda - m)}} \begin{pmatrix} k_x - ik_y \\ \lambda - m \end{pmatrix}$$

$$\boxed{\lambda(k) = \sqrt{k^2 + m^2}}$$

$$\left|u^-(k)\right\rangle = \frac{1}{\sqrt{2\lambda(\lambda + m)}} \begin{pmatrix} -k_x + ik_y \\ \lambda + m \end{pmatrix}$$

Berry connection

$$\boxed{A_\mu = \sum_a^{filled} \left\langle u^a \left| \frac{d}{dk_\mu} \right| u^a \right\rangle}$$

$$A_x(k,m) = \frac{ik_y}{2\lambda(\lambda + m)}$$

$$A_y(k,m) = \frac{-ik_x}{2\lambda(\lambda + m)}$$

Dirac Model of IQHE

Berry curvature

$$F_{xy}(k,m) = \partial_{k_x} A_y - \partial_{k_y} A_x = -\frac{im}{2\lambda^3}$$

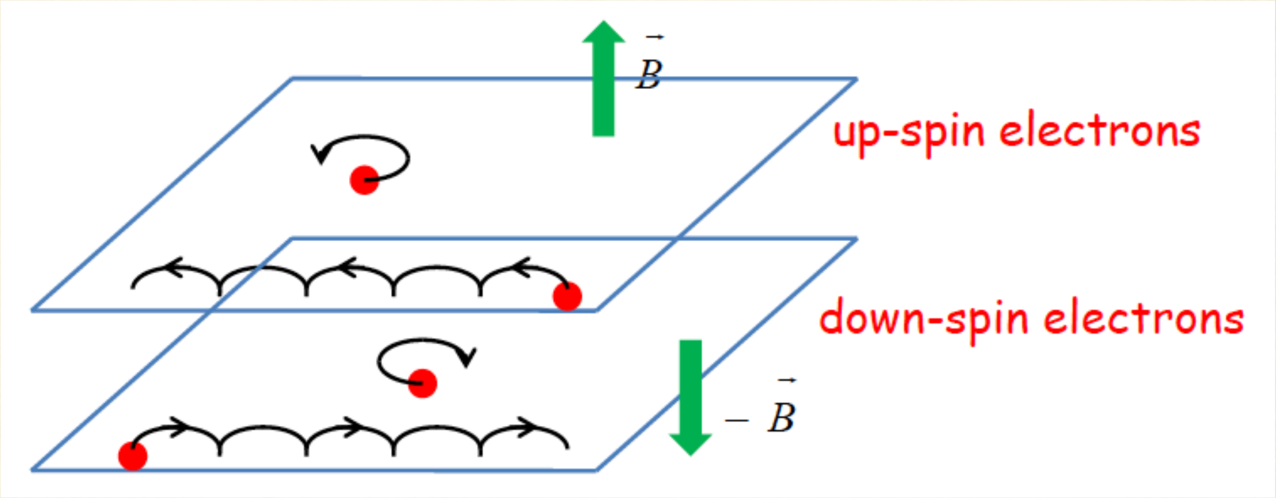
Chern number

$$C_1[F] = \frac{i}{2\pi}\int d^2k F_{xy} = \frac{i}{2\pi}\int d^2k \frac{-im}{2\lambda^3} = \frac{1}{2}\frac{m}{|m|}$$

UV physics [e.g., m → m(k) = m – Ck^2, or Pauli-Villars regularization]

$$\sigma_{xy} = \begin{cases} 1, & \mathrm{sgn}(m) = \mathrm{sgn}(C) \\ 0, & \mathrm{sgn}(m) = -\mathrm{sgn}(C) \end{cases}$$

Dirac Model of QSHE



$$H = \begin{pmatrix} -i\partial_x\sigma_x - i\partial_y\sigma_y + m\sigma_z & a\sigma_0 \\ a\sigma_0 & -i\partial_x\sigma_x - i\partial_y\sigma_y - m\sigma_z \end{pmatrix}$$

a = 0: Two copies of the Dirac model of IQHE.

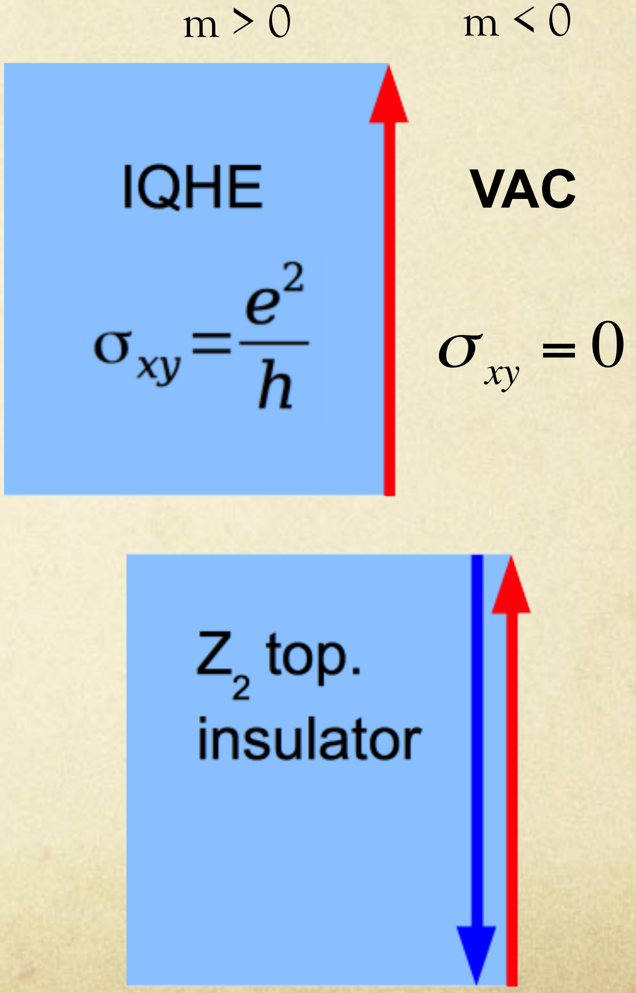
Field a breaks spin-rotational symmetry.

$$\hat{\Theta} = \left(i\sigma_y \otimes \tau_x\right)K$$

Topological Insulators and Superconductors

- Integer quantum Hall system is an example of topological insulators, which have gapped bulk but gapless edge modes (skipping orbits).
- A new type of topological insulator has been introduced in so called quantum spin Hall effect.
- In fact, one can classify topological insulators and topological superconductors by time-reversal, particle-hole, and chiral symmetries. There are exactly five classes of topological insulators in each dimension.

See Schnyder et al., Phys. Rev. B (2008);
also A. Kitaev (arXiv:0901.2686)



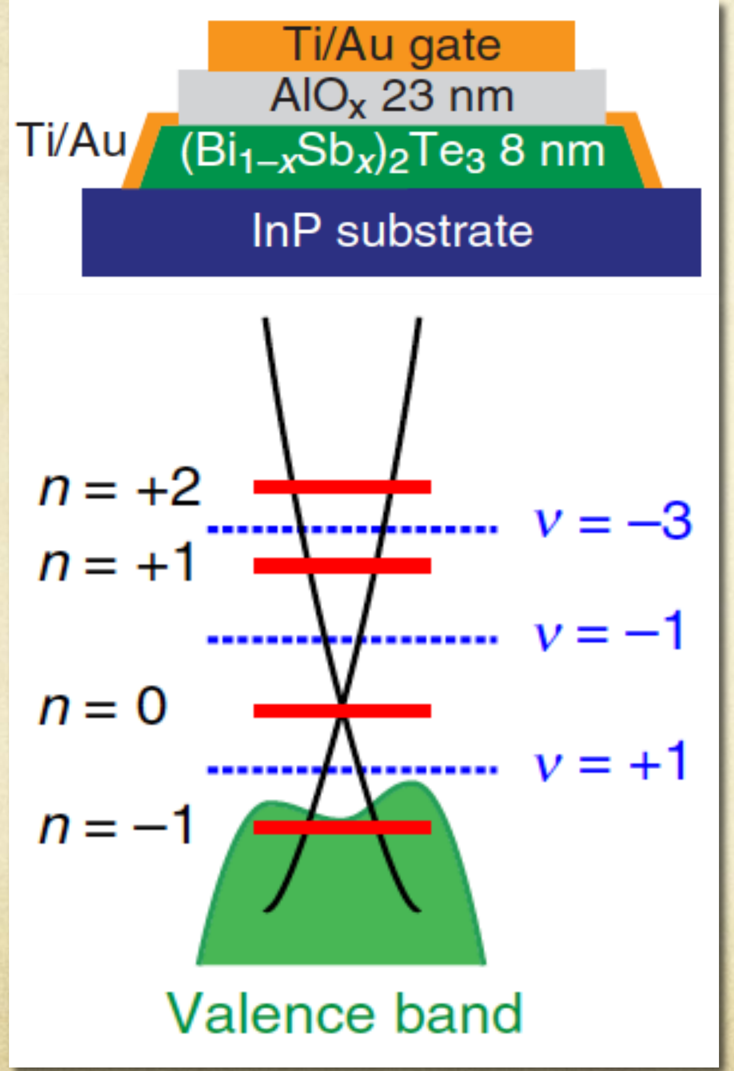
The Ten-Fold Way

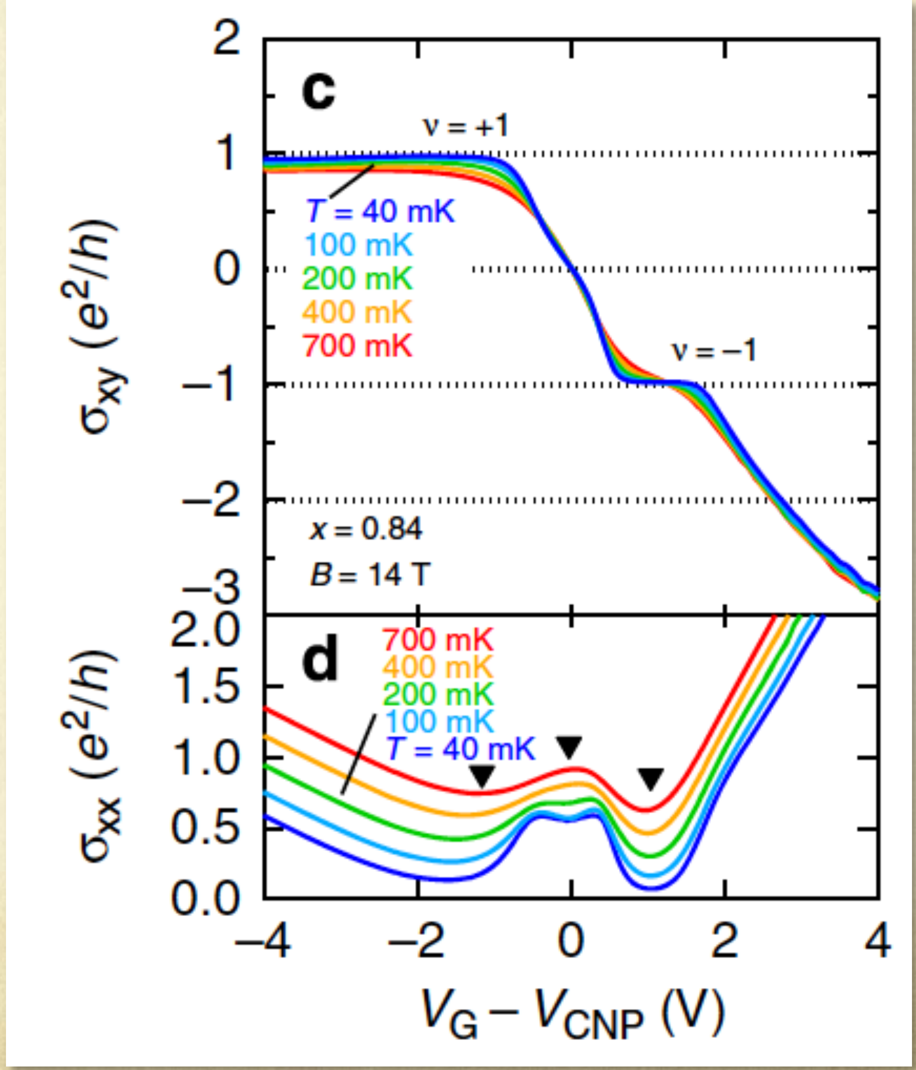
Cartan nomenclature	TRS	PHS	SLS	Hamiltonian	$d=1$	$d=2$	$d=3$
AIII (chiral unit.)	0	0	1	$U(N+M)/U(N)\times U(M)$	$\mathbf{Z}$	-	$\mathbf{Z}$
A (unitary)	0	0	0	$U(N)$	-	$\mathbf{Z}$	-
BDI (chiral orthog.)	$+1$	$+1$	1	$SO(N+M)/SO(N)\times SO(M)$	$\mathbf{Z}$	-	-
D	0	$+1$	0	$SO(2N)$	$\mathbf{Z}_2$	$\mathbf{Z}$	-
DIII	-1	$+1$	1	$SO(2N)/U(N)$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}$
AII (symplectic)	-1	0	0	$U(2N)/Sp(2N)$	-	$\mathbf{Z}_2$	$\mathbf{Z}_2$
CII (chiral sympl.)	-1	-1	1	$Sp(2N+2M)/Sp(2N)\times Sp(2M)$	$\mathbf{Z}$	-	$\mathbf{Z}_2$
C	0	-1	0	$Sp(2N)$	-	$\mathbf{Z}$	-
CI	$+1$	-1	1	$Sp(2N)/U(N)$	-	-	$\mathbf{Z}$
AI (orthogonal)	$+1$	0	0	$U(N)/O(N)$	-	-	-

See Schnyder et al., Phys. Rev. B (2008); also A. Kitaev (arXiv:0901.2686)

3D Topological Insulators

Yoshimi et al. (2015)





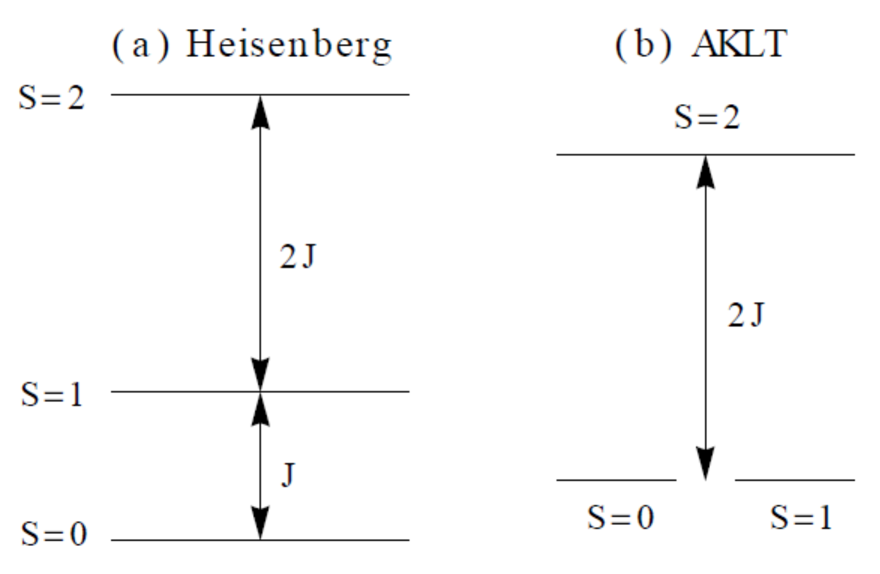
Spin-1 AKLT State

$$H = \frac{1}{2}\sum_i \left[\vec{S}_i \cdot \vec{S}_{i+1} + \frac{1}{3}\left(\vec{S}_i \cdot \vec{S}_{i+1}\right)^2 + \frac{2}{3} \right] = \sum_i P_2(i, i+1)$$

I. Affleck, T. Kennedy, E. H. Lieb, and H. Tasaki, Commun. Math. Phys. 115, 477 (1988)

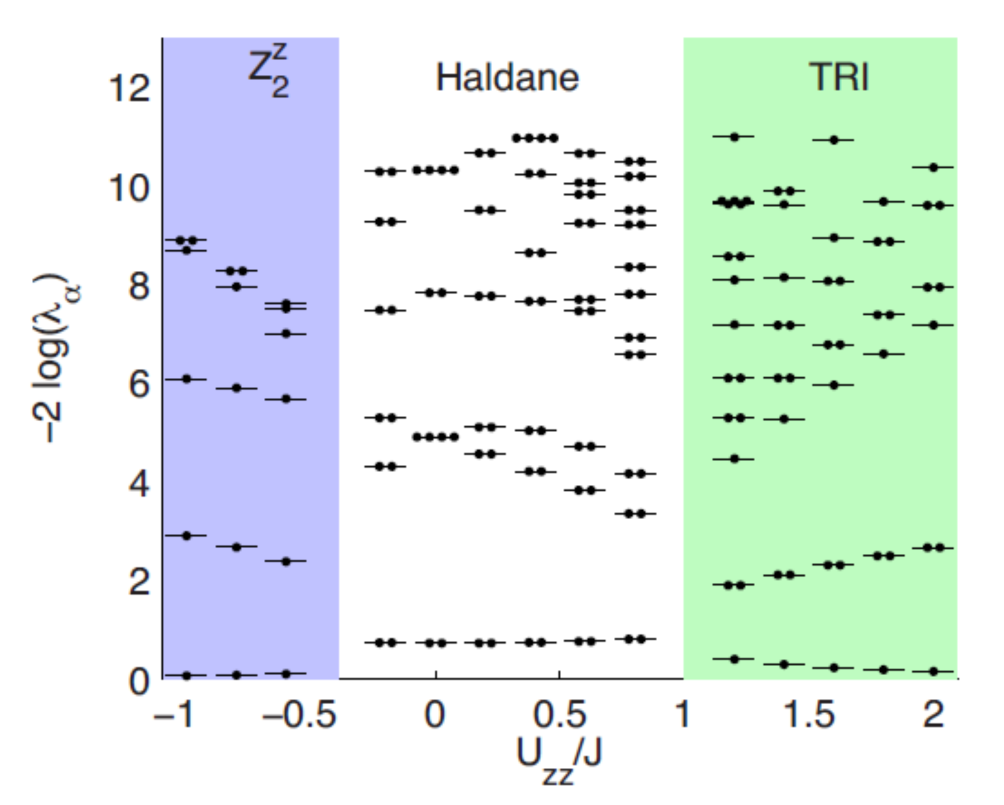
Describes the Haldane gapped phase of spin-1

$$H = J\sum_i S_i \cdot S_{i+1}$$



Symmetry Protected Topological Phase

$$H_0 = J\sum_j \vec{S}_j \cdot \vec{S}_{j+1} + B_x \sum_j S_j^x + U_{zz} \sum_j (S_j^z)^2$$



Pollmann et al., PRB (2010)

The Haldane phase is characterized by a double degeneracy of the entire entanglement spectrum.

The degeneracy, or the Haldane phase is protected by any of the following symmetries: time-reversal symmetry, dihedral symmetry D_2, or spatial inversion symmetry.

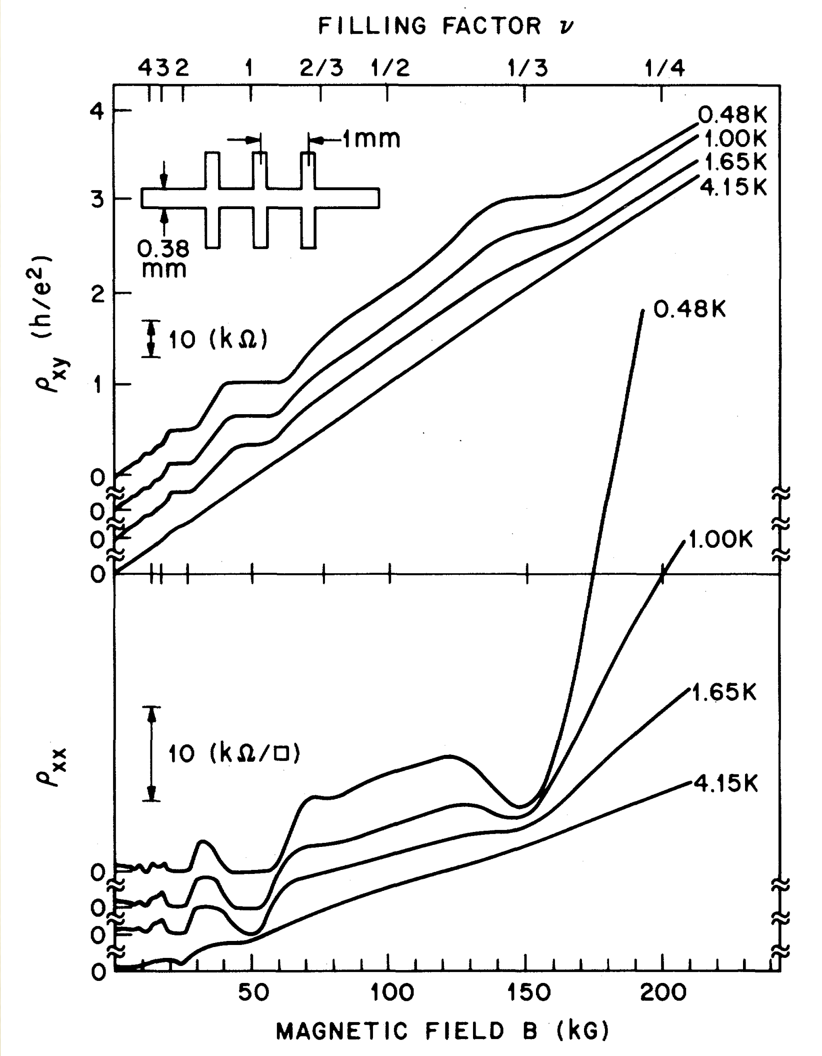
Before the Break, …

- Outline of Lecture 2
  - AKLT state and Introduction to MPS
  - iTEBD and its applications
  - MPS of the Laughlin state

- Subscribe PhysicsChat
  - Type “Nanjing2016”

Install Anaconda:
https://www.continuum.io/downloads

FQHE (1982)



Daniel C. Tsui

Horst L. Störmer

$$R_H = \frac{V_H}{I} = \frac{h}{\nu e^2}$$

$$R = \frac{V_{xx}}{I}$$

Fractional filling factor: interaction important!

Ground State Degeneracy

- One can use the many-body wave function of a FQH state to calculate the corresponding topological Chern number, which should be an integer, not a fraction!

Ground state must be degenerate in the thermodynamic limit (Wen & Niu, 90)

Shared by 3 states, Hall conductance 1/3. FQHE!

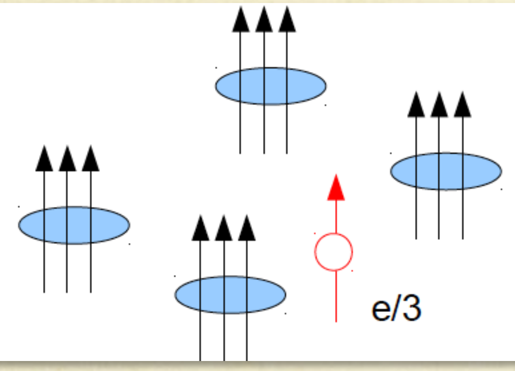
e.g., $\nu = 1/3$

$$E_b \propto e^{-L(m^* \Delta)^{1/2}}$$

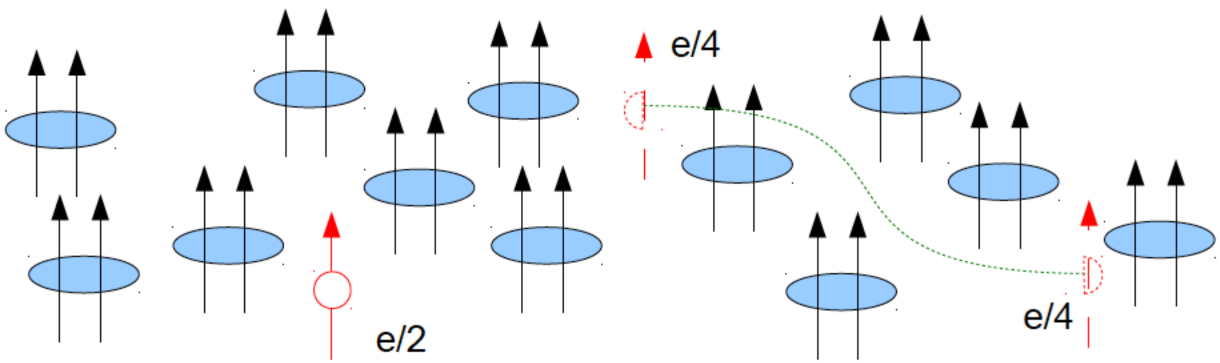
$$\sum_{m_0}^{3} C_1(m_0) = 1$$

Fractionalized Excitations

- Abelian Laughlin state (ν = 1/3)

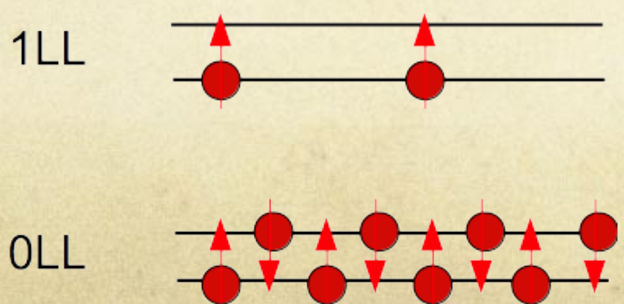


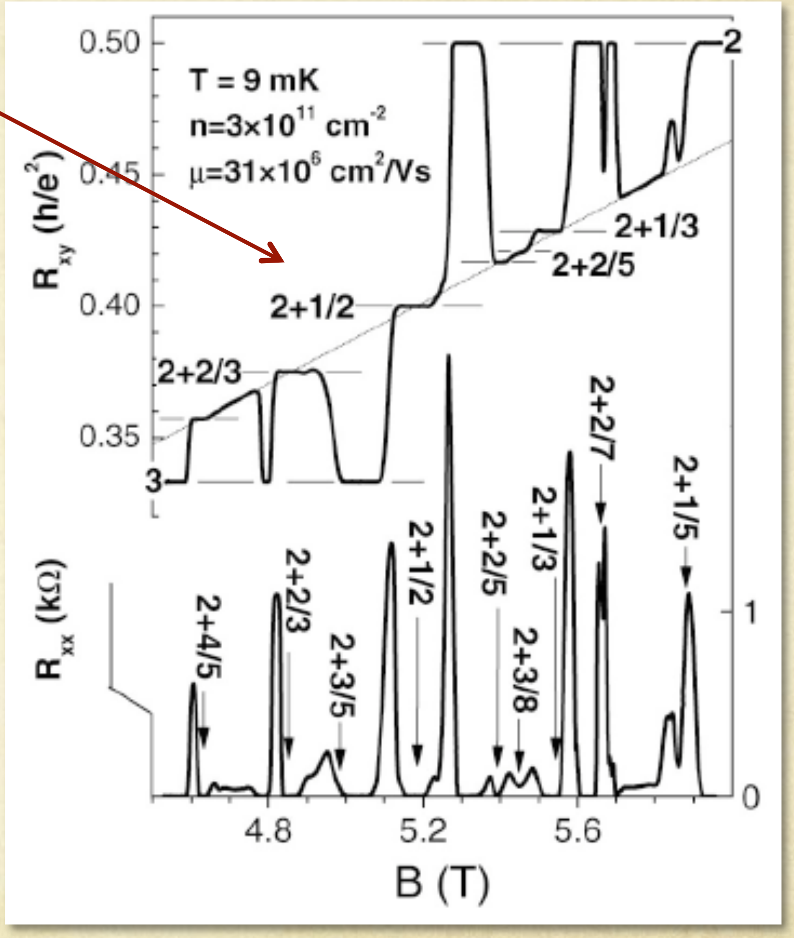
- Non-Abelian Moore-Read state (ν = 5/2)



FQH Effect at $\nu = 5/2$

- Discovered by R. Willett, 1987
- Spin-polarized wavefunction based on Ising conformal field theory, Moore & Read, 1991
- Numerical verification, Morf, 1998; Rezayi & Haldane, 2000
- Proposal of topologically protected qubits, Das Sarma, Freedman & Nayak, 2005





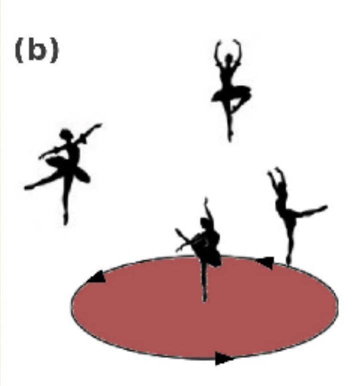
Xia et al., PRL (2004)

Electron Dancing Patterns

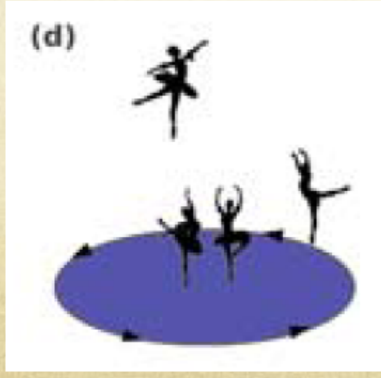
- 1/3 Langhlin state $z = x + iy$

$$H_{hardcore} = \sum_{i<j}^{N} \partial_i^2 \delta^2 (z_i - z_j)$$

$$\Psi_L = \prod_{i<j} (z_i - z_j)^2 \prod_{i<j} (z_i - z_j) e^{-\sum_i |z_i|^2/4}$$



- 1/2 Moore-Read state



$$H_{3B} = -\sum_{i<j<k}^{N} S_{ijk} \left\{ \nabla_i^2 \nabla_j^4 \delta^2 (z_i - z_j) \delta^2 (z_i - z_k) \right\}$$

$$\Psi_{Pf} = Pf\left(\frac{1}{z_i - z_j}\right) \prod_{i<j} (z_i - z_j) \prod_{i<j} (z_i - z_j) e^{-\sum_i |z_i|^2/4}$$

The Moore-Read State

- Moore-Read state (Moore & Read, 1991)

$$\Psi_{Pf} = \langle \psi_e(z_1) \cdots \psi_e(z_N) \rangle = Pf\left(\frac{1}{z_i - z_j}\right) \prod_{i<j} (z_i - z_j)^m$$

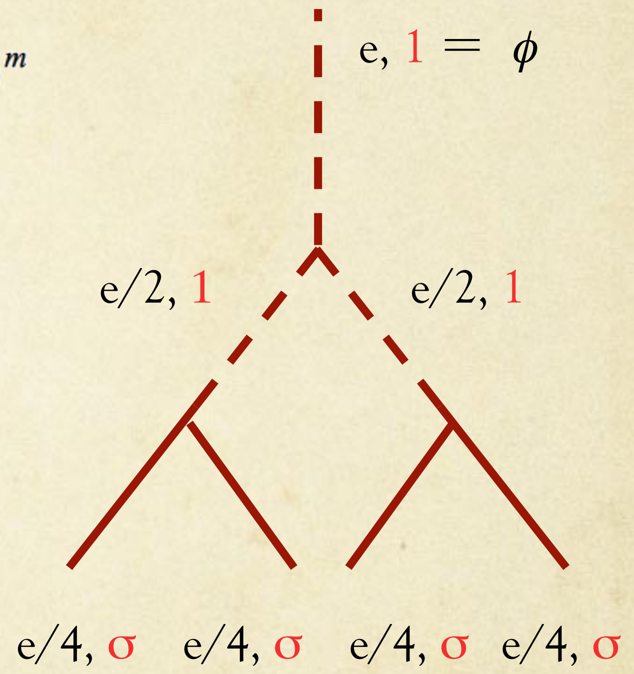
$$Pf\left(\frac{1}{z_i - z_j}\right) = A\left(\frac{1}{z_1 - z_2}\frac{1}{z_3 - z_4}\cdots\frac{1}{z_{N-1} - z_N}\right)$$

- Abelian quasiholes (charge e/2)

$$\prod_i (z_i - \xi_1)(z_i - \xi_2) Pf\left(\frac{1}{z_i - z_j}\right) \prod_{1 \le i < j \le N} (z_i - z_j)^2$$

- Non-Abelian quasiholes (charge e/4)

$$\Psi_{(12)(34)} = Pf\left(\frac{(z_i - \xi_1)(z_i - \xi_2)(z_j - \xi_3)(z_j - \xi_4) + i \Leftrightarrow j}{z_i - z_j}\right) \prod_{1 \le i < j \le N} (z_i - z_j)^2$$



Degenerate Ground States

- Internal structure of four non-Abelian anyons

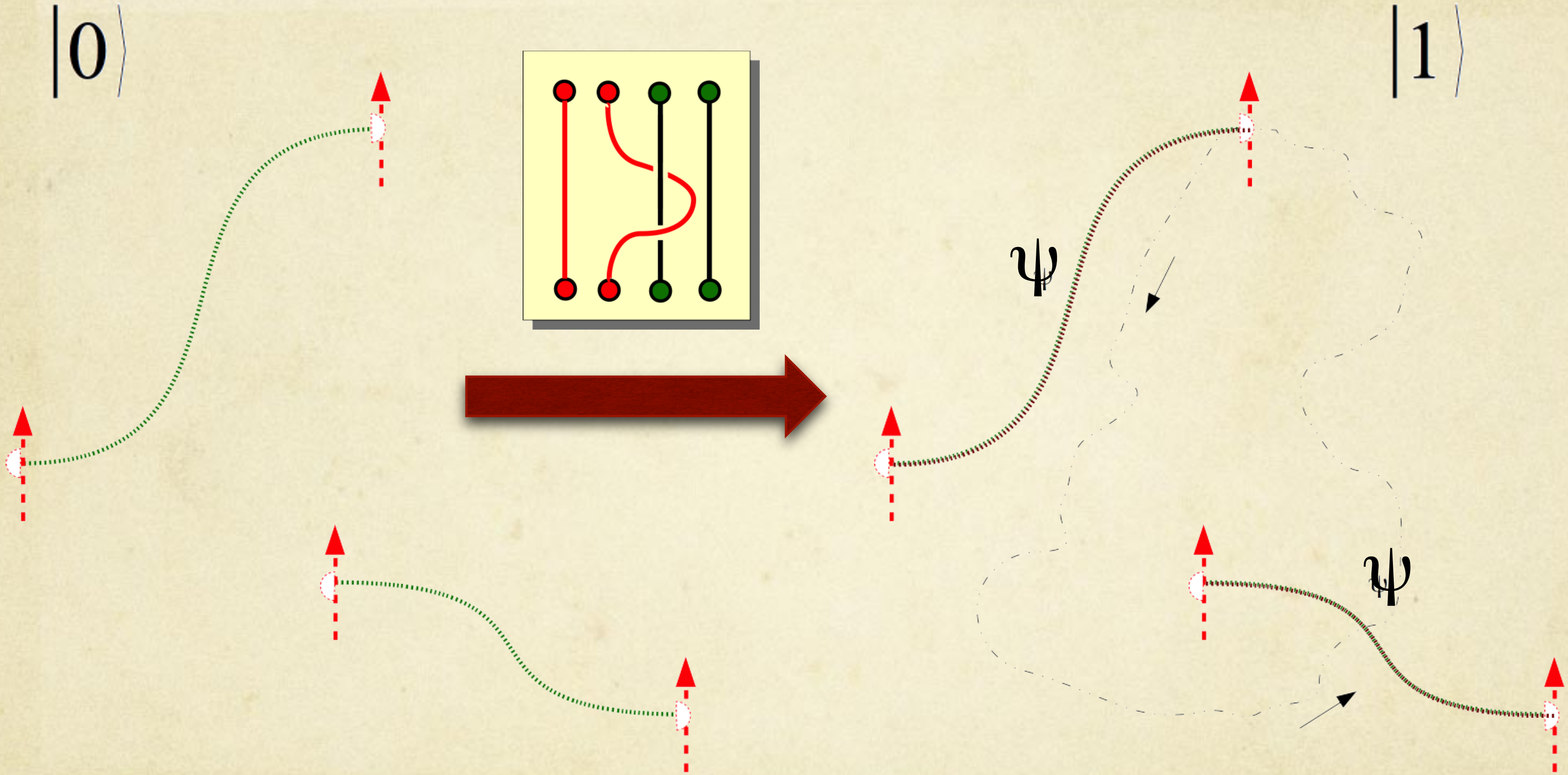
$$\Psi_{(12)(34)} = Pf\left(\frac{(z_i-\xi_1)(z_i-\xi_2)(z_j-\xi_3)(z_j-\xi_4)+i\Leftrightarrow j}{z_i-z_j}\right)\prod_{1\leqslant i<j\leqslant N}(z_i-z_j)^2$$

$$\Psi_{(13)(24)} = Pf\left(\frac{(z_i-\xi_1)(z_i-\xi_3)(z_j-\xi_2)(z_j-\xi_4)+i\Leftrightarrow j}{z_i-z_j}\right)\prod_{1\leqslant i<j\leqslant N}(z_i-z_j)^2$$

$$\Psi_{(14)(23)} = Pf\left(\frac{(z_i-\xi_1)(z_i-\xi_4)(z_j-\xi_2)(z_j-\xi_3)+i\Leftrightarrow j}{z_i-z_j}\right)\prod_{1\leqslant i<j\leqslant N}(z_i-z_j)^2$$

$$\Psi_{(12)(34)}-\Psi_{(13)(24)}=(1-x)\left(\Psi_{(12)(34)}-\Psi_{(14)(23)}\right) \qquad x=\frac{(\xi_1-\xi_2)(\xi_3-\xi_4)}{(\xi_1-\xi_3)(\xi_2-\xi_4)}$$

Braiding

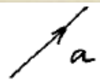


$$\sigma \times \sigma = 1 + \psi$$

Outline

- Introduction
- Examples of topological phases
- Implementation of topological quantum computation
  - Topological quantum field theory
  - Non-Abelian anyons in the Moore-Read state
  - Topological p-wave superconductor
  - Majorana zero modes at the ends of quantum wires
- Summary

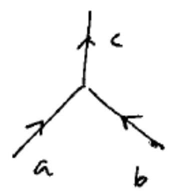
Diagrams

– Anyon a : 

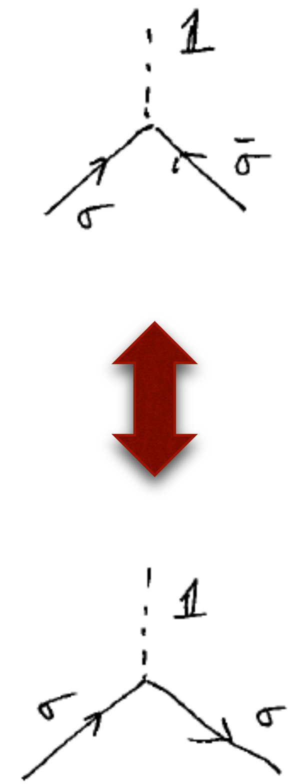
– Antiparticle $\bar{a}$: 

= 

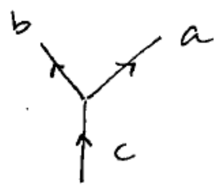
– Fusion $a \times b = c \; (+ \cdots)$

$\left(\frac{d_c}{d_a d_b}\right)^{1/4}$ 

$= \langle ab; c|$

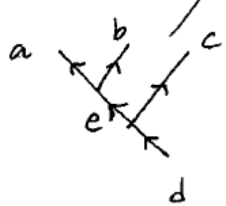


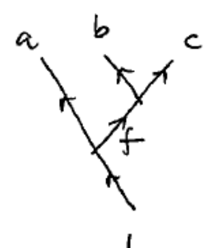
– Braiding

$= R^{ab}_{c}$ 

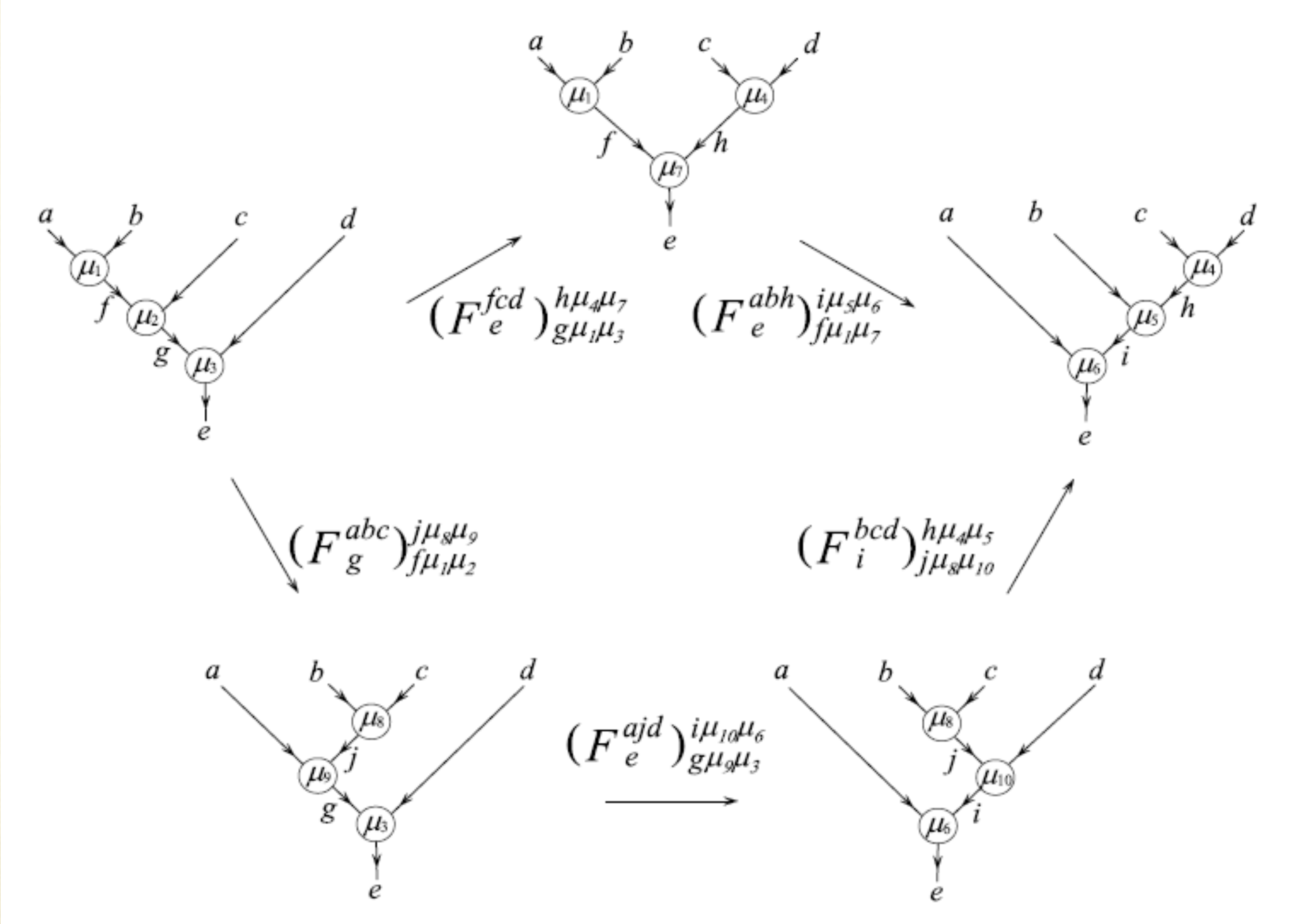
phase

– Associativity

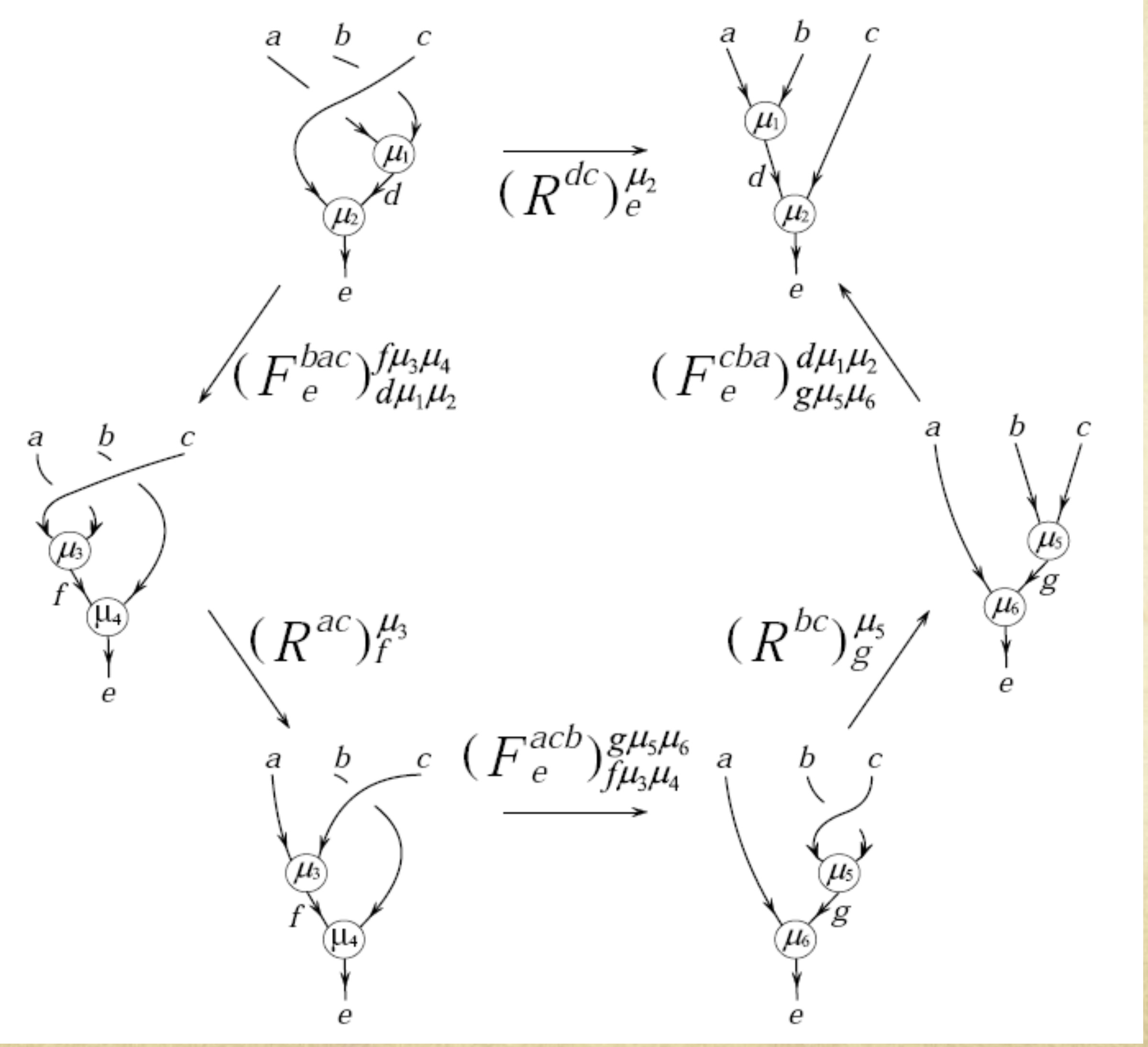


$= \sum_f \left[F^{abc}_{d}\right]_{ef}$ 

Pentagon Equation



Hexagon Equation



Theory of Anyons: TQFT

- A model of anyons is a theory of a two-dimensional medium with a mass gap, where the particles carry locally conserved charges.
  - A finite *label set* $\{a,b,c,\ldots\}$
  - *Fusion rules* $a \times b = \Sigma_c N_{ab}^{c}\, c$
  - The *F-matrix* (expressing associativity of fusion)
  - The *R-matrix* (braiding rules)

Ising anyon model:

$\{1, \sigma, \psi\}$

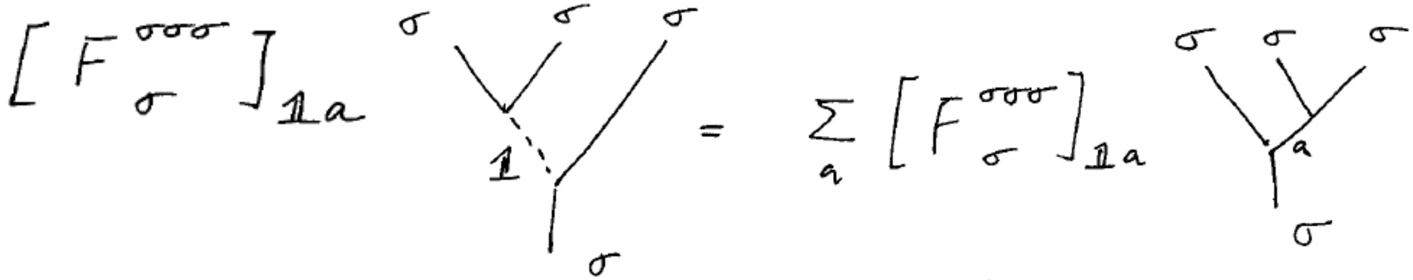
$\sigma \times \sigma = 1 + \psi$
$\psi \times \psi = 1$
$\psi \times \sigma = \sigma \times \psi = \sigma$

$$F = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

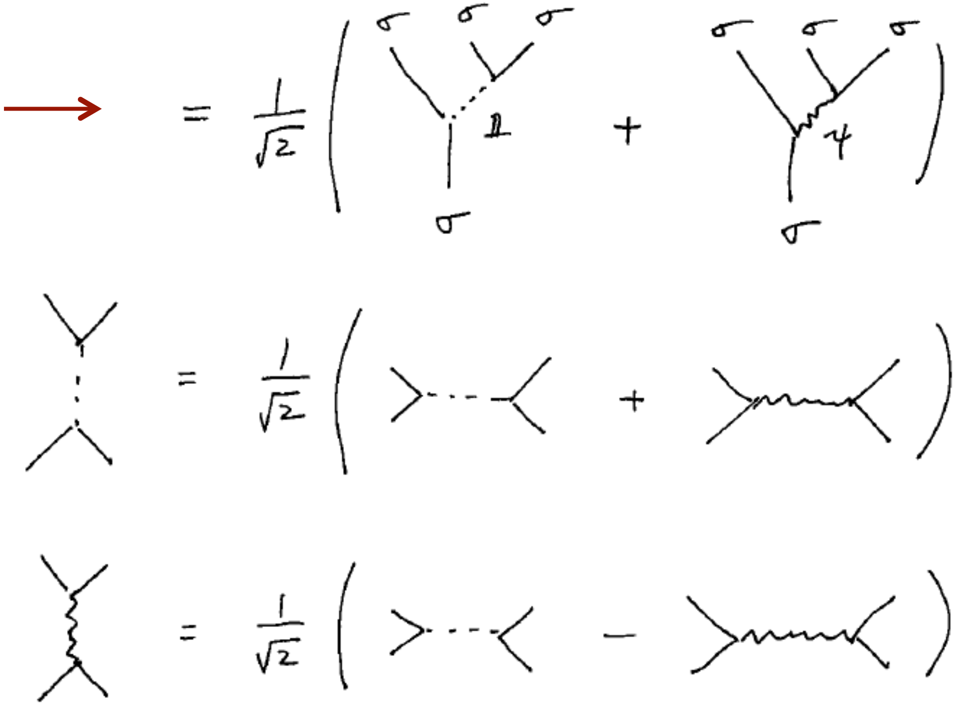
$$R = \begin{pmatrix} e^{-i\pi/8} & 0 \\ 0 & e^{3i\pi/8} \end{pmatrix}$$

F & R satisfy self-consistency equations: the pentagon and hexagon equations.

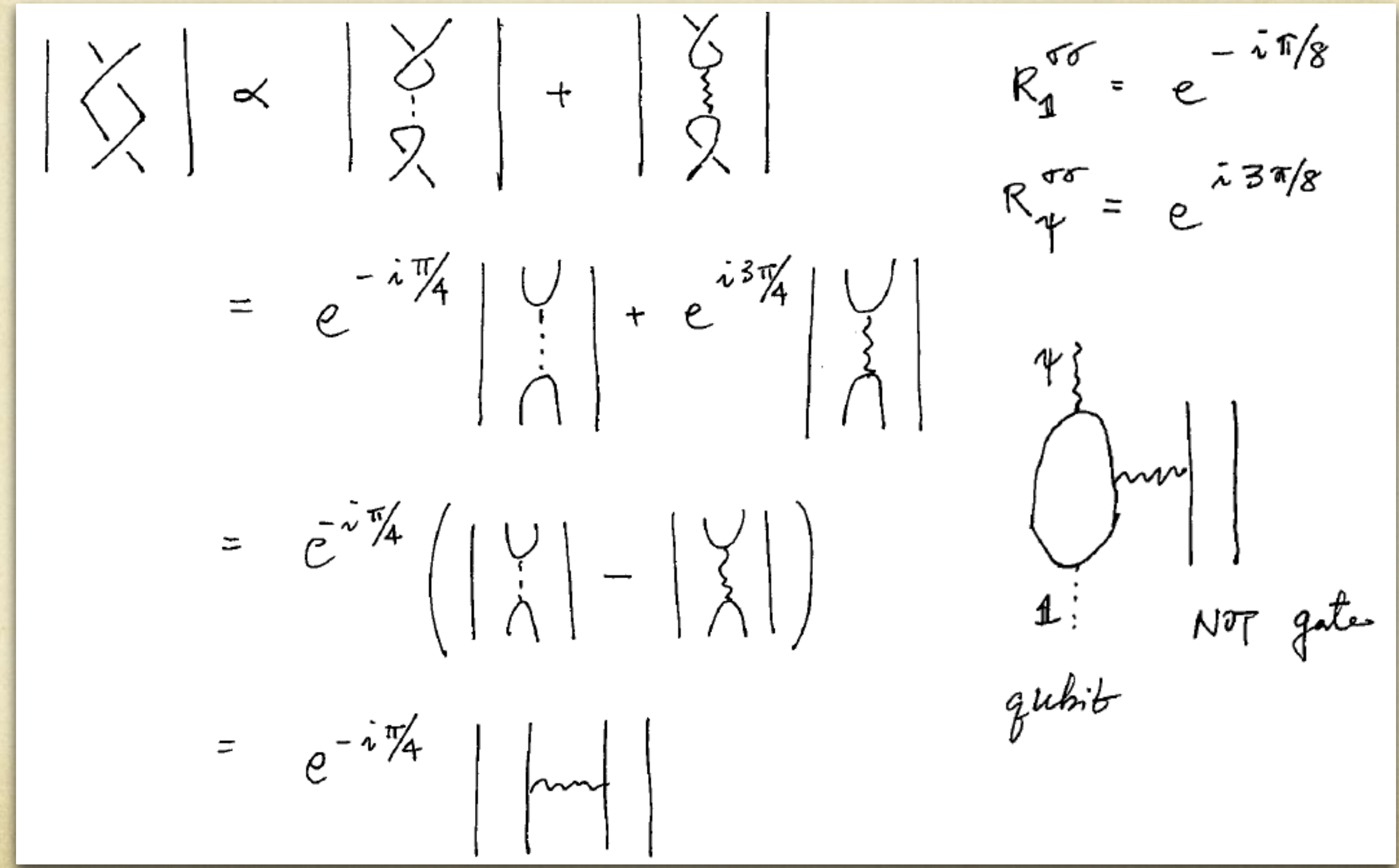
Application of F-Matrix



$$F = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$



NOT Gate

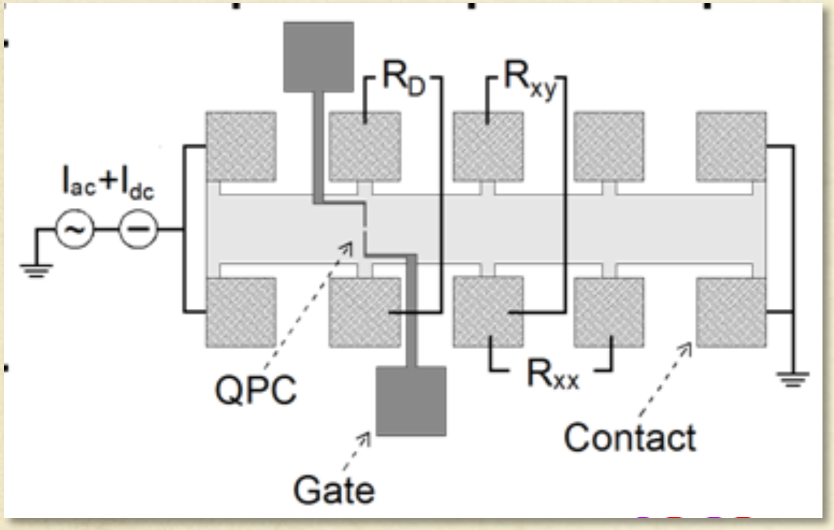


∝
+
= $e^{-i\pi/4}$ + $e^{i3\pi/4}$
= $e^{-i\pi/4}$ (−)
= $e^{-i\pi/4}$
$R_1^{\sigma\sigma} = e^{-i\pi/8}$
$R_\psi^{\sigma\sigma} = e^{i3\pi/8}$
ψ
1
NOT gate
qubit

5/2 State: The Road Map

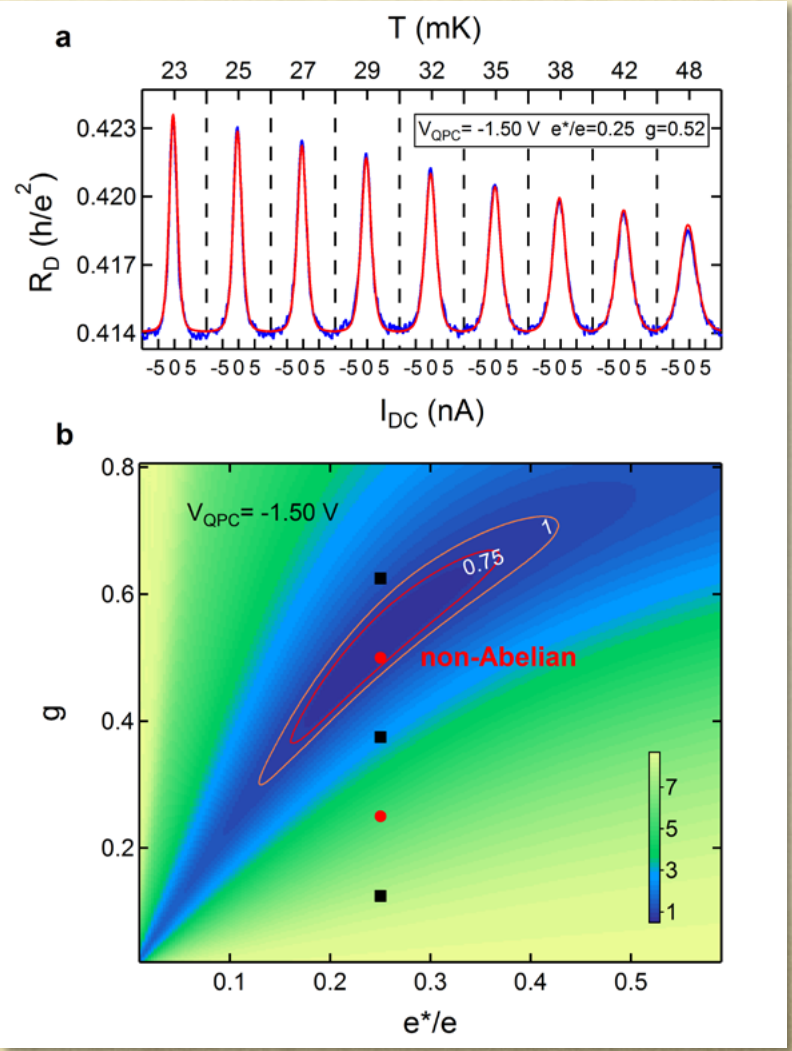
- ✓ Determine the spin-polarization of the $\nu = 5/2$ state.
- ✓ Study the 5/2 state in gated samples, which will be necessary for interferometry and, ultimately, quantum computation.
- ✓ Study quasiparticle tunneling at quantum point contacts.
- ✓ Determine quasiparticle electrical charge through antidot and noise experiments.
- ➢ Determine quasiparticle braiding statistics through interferometry.
- ○ Construct a 5/2 qubit and NOT gate.

Lin Group (2015)



$$g_T(T, I_{DC}) = AT^{(2g-2)}F(g, \frac{e^* I_{DC} R_{XY}}{k_B T})$$

Two-parameter fitting of the tunneling conductance can reveal information on quasiparticle charge e* and interaction parameter g.

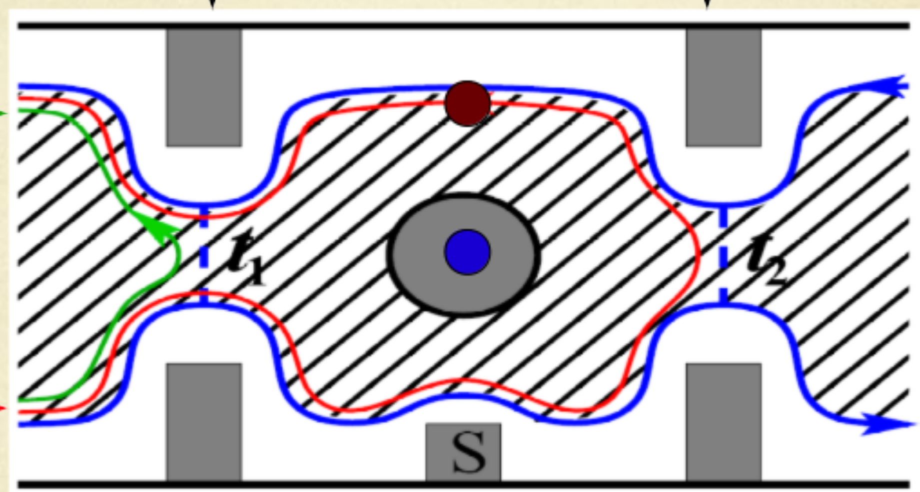


Quantum Hall Interferometer

Gates controlling the strength of tunneling

path via point contact 1

path via point contact 2



edge of the ½ droplet

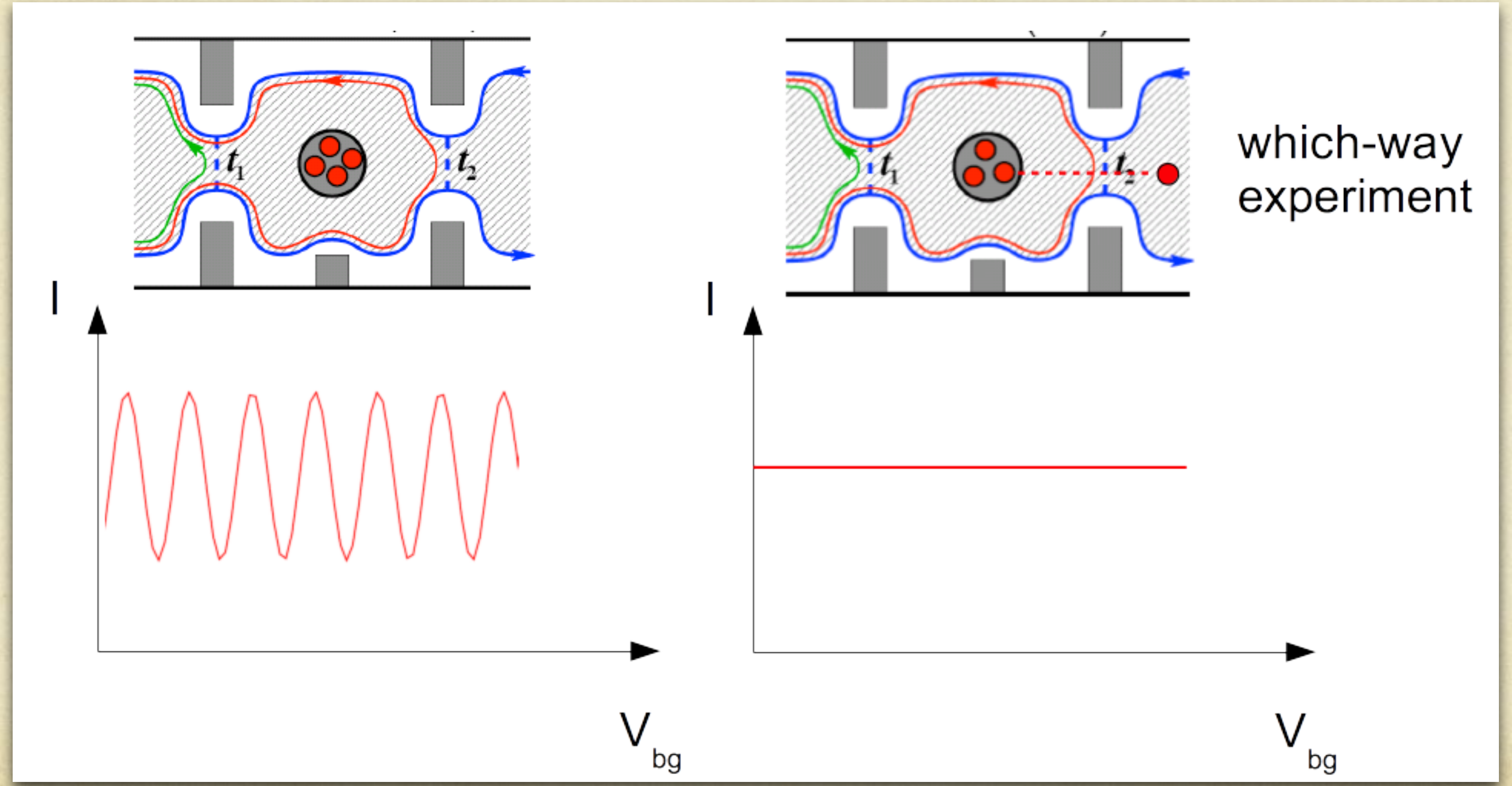
edges of the filled Landau levels not included

Side gate controlling the number of quasiparticles on the central antidot

$$G \propto \left| t_1 U_1 + t_2 U_2 |\Psi\rangle \right|^2 = |t_1|^2 + |t_2|^2 + 2\,\Re\left\{ t_1^* t_2 e^{i\phi} \langle \Psi | M_n | \Psi \rangle \right\}$$

Expected Experimental Signature

Odd-even effect: Stern & Halperin (06); Bonderson, Kitaev & Shtengel (06)



Even number of non-Abelian qps inside the interference loop

Odd number of non-Abelian qps inside the interference loop

The Tale of Two Anyons

$$\Psi^{e/4}_{qh} = \sigma\, e^{i\phi/2\sqrt{2}}$$

Most relevant.
Charge & neutral components.

$$\Psi^{e/2}_{qh} = e^{i\varphi/\sqrt{2}},\ \cancel{\psi\, e^{i\varphi/\sqrt{2}}}$$

Irrelevant to inter-edge tunneling in RG sense

Less relevant but relevant
Charge component only!

$$\sigma \times \sigma = 1 + \psi \quad \text{(Ising/Majorana)}$$

$$I_{12} \propto \sum_q s_q |\Gamma_1||\Gamma_2| e^{-|x_1 - x_2|/L_\phi} \cos\left(2\pi \frac{q}{e}\frac{\Phi}{\Phi_0} + \phi_q + arg\left(\Gamma_1 \Gamma_2^*\right)\right)$$

tunneling amplitude

favors e/4 qps

coherence length due to thermal smearing

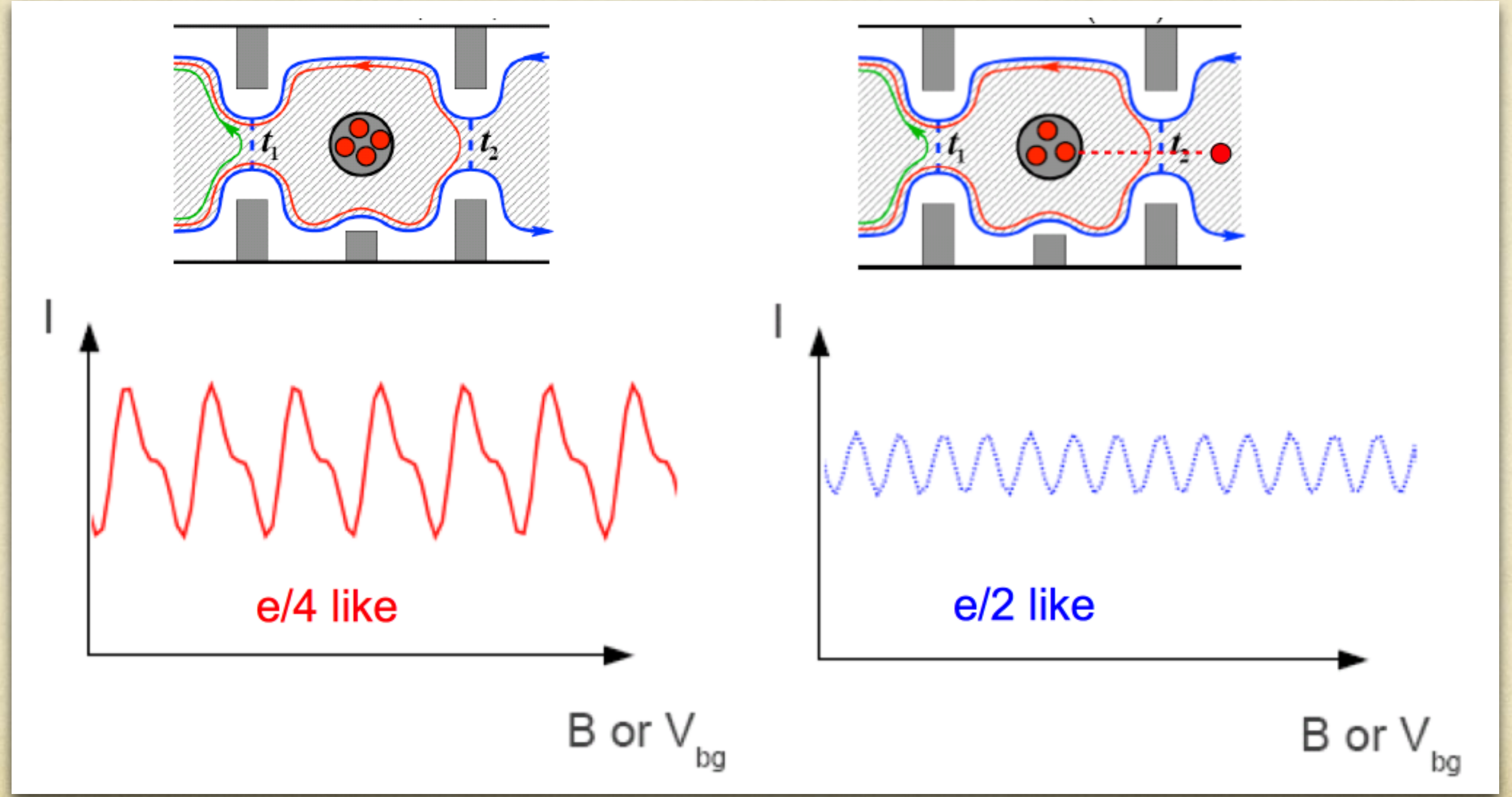
$$L_\phi = \frac{1}{2\pi k_B T}\left(\frac{g_c}{v_c} + \cancel{\frac{g_n}{v_n}}\right)^{-1}$$

favors e/2 qps

Refined Expectation

Background Abelian signal:

XW, Hu, Rezayi & Yang, PRB (2008)



Coherence length ~ 1 μm

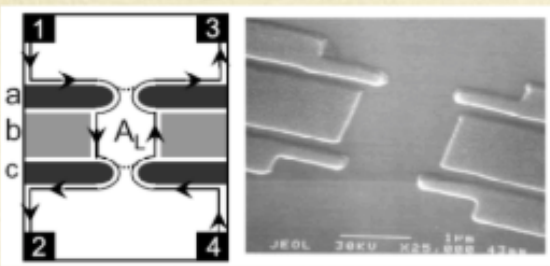
Willett et al., PNAS (2009)

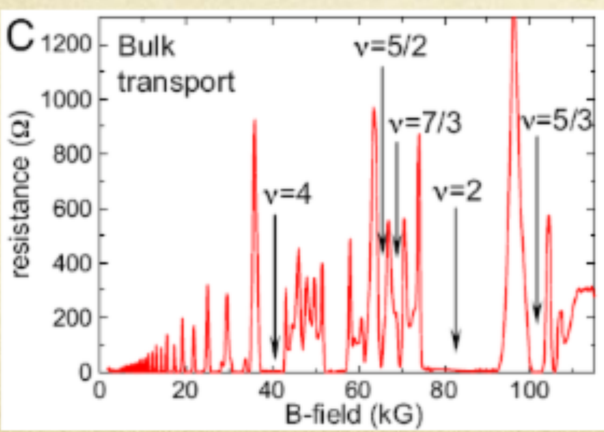
Measurement of filling factor 5/2 quasiparticle interference with observation of charge e/4 and e/2 period oscillations

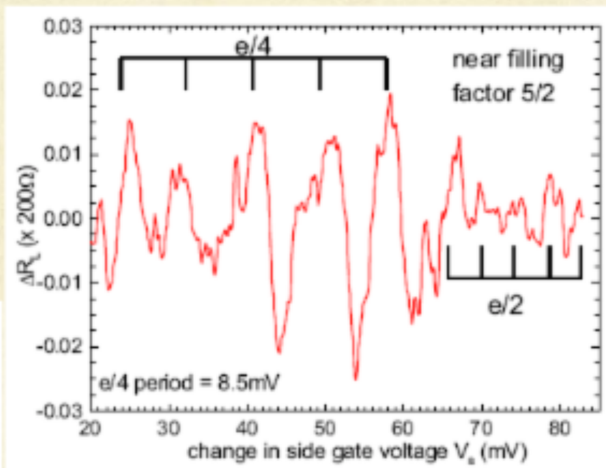
R. L. Willett[1], L. N. Pfeiffer, and K. W. West

Physical Sciences Research, Bell Laboratories, Alcatel-Lucent, 600 Mountain Avenue, Murray Hill, NJ 07974

Edited by W. F. Brinkman, Princeton University, Princeton, NJ, and approved March 30, 2009 (received for review December 10, 2008)







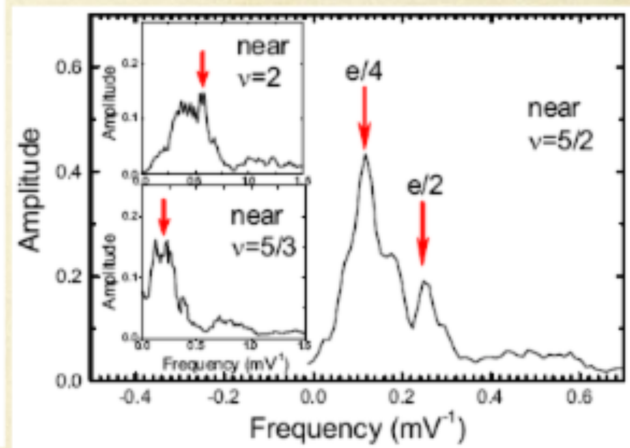
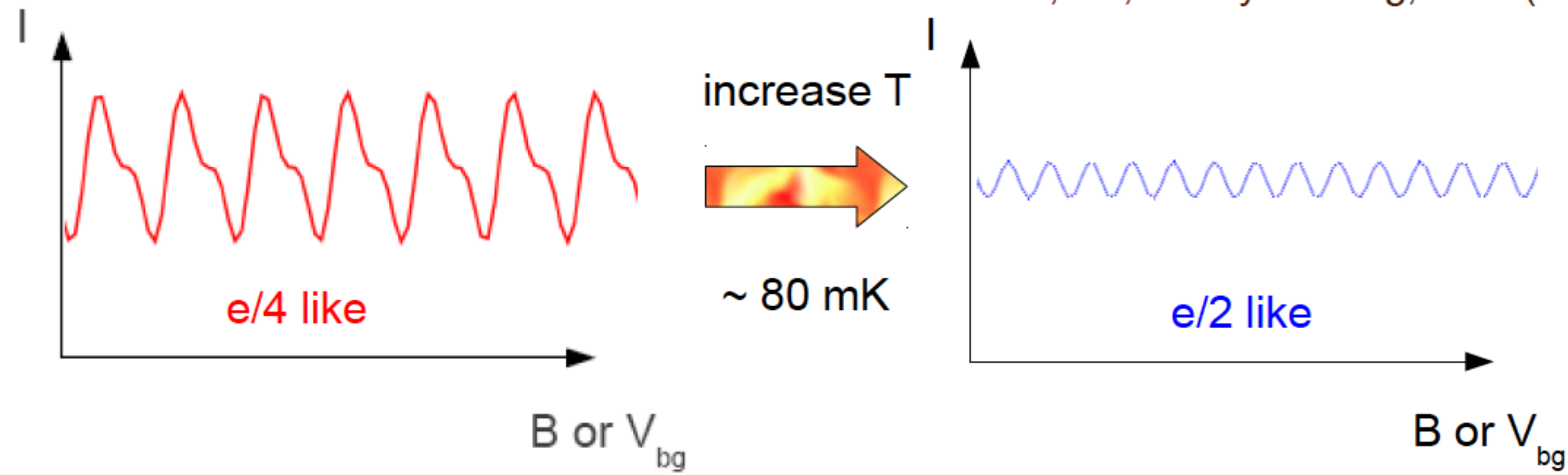
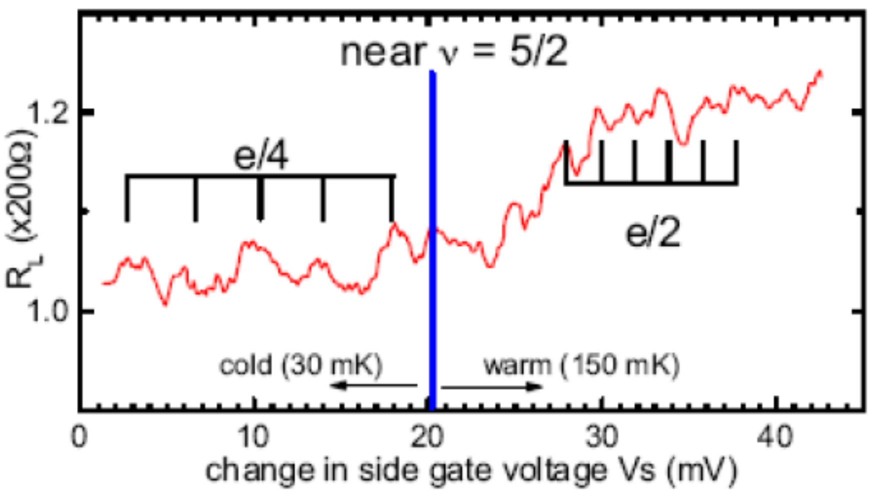
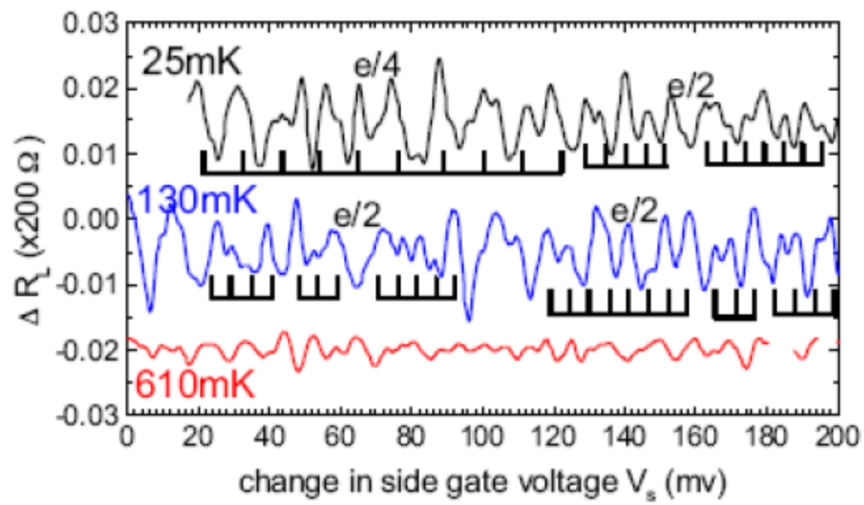


Fig. 3. A–B interferometric measurements at 5/2 filling factor displaying oscillations with periods corresponding to e/4 and e/2. R_L is measured for side-gate sweeps at filling factor $\nu = 5/2$ using similar measurements at $\nu = 2$ and 5/3 or 7/3 as metrics. Left and right

XW, Hu, Rezayi & Yang, PRB (2008)



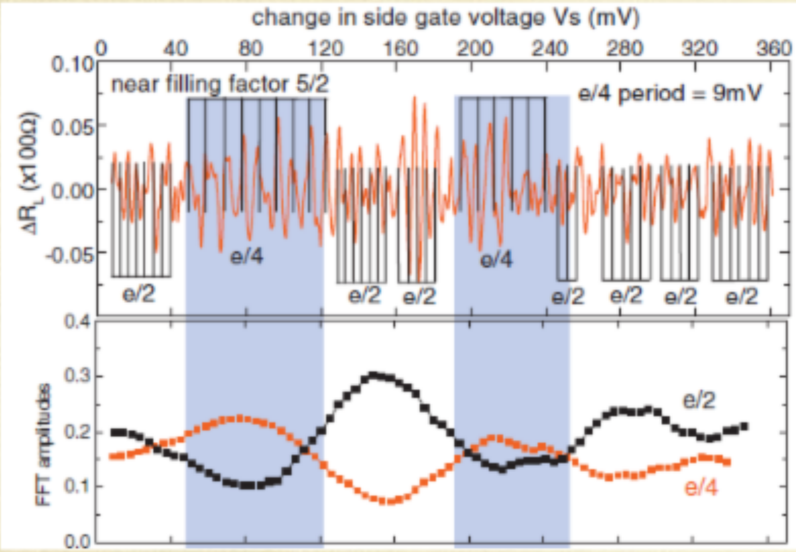


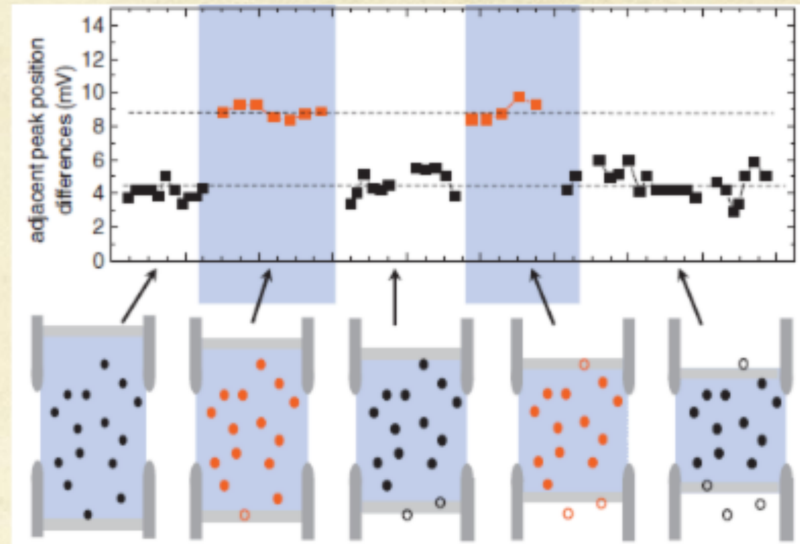


period lines in the swept side-gate data. (*C*) Data indicate temperature dependence of e/4 and e/2 oscillations: e/2 oscillations may be made more prevalent with an increase in temperature. The temperature of the sample was taken from

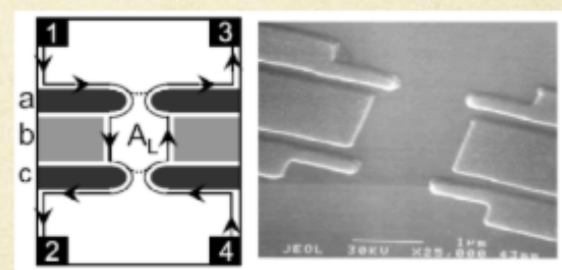
Willett et al., PNAS (2009)

Willett et al., PRB (2010)

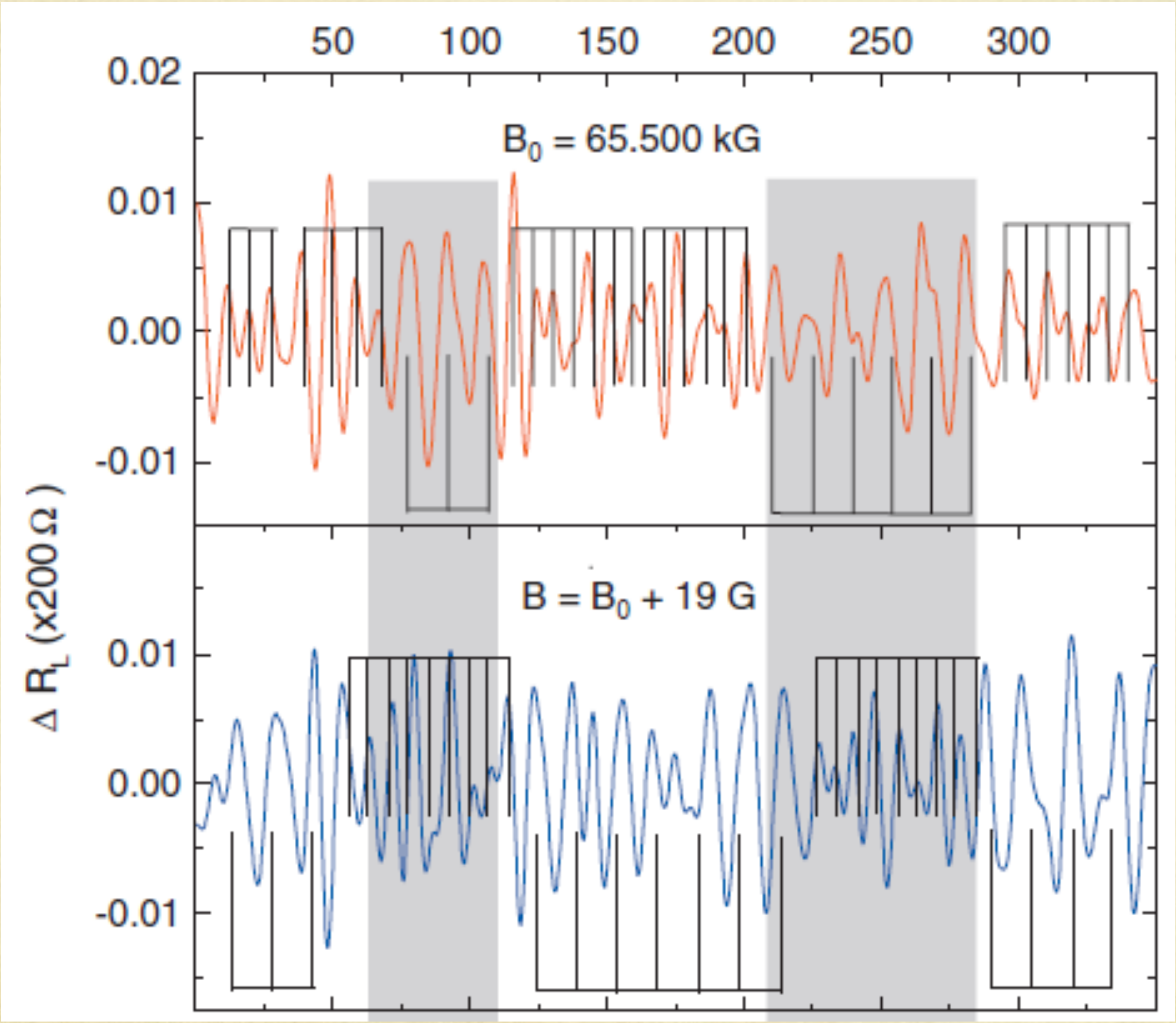


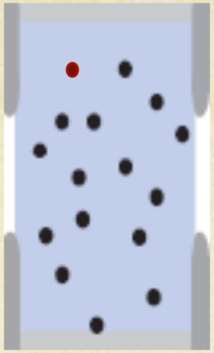


On interference & thermal decoherence:
Bishara & Nayak, Phys. Rev. B 77, 165302 (2008)
XW, Hu, Rezayi & Yang, Phys. Rev. B 77, 165316 (2008)



On tunneling amplitude:
Bishara, Bonderson, Nayak, Shtengel & Slingerland,
Phys. Rev. B 80, 155303 (2009)
Chen, Hu, Yang, Rezayi & XW, Phys. Rev. B 80, 235305 (2009)



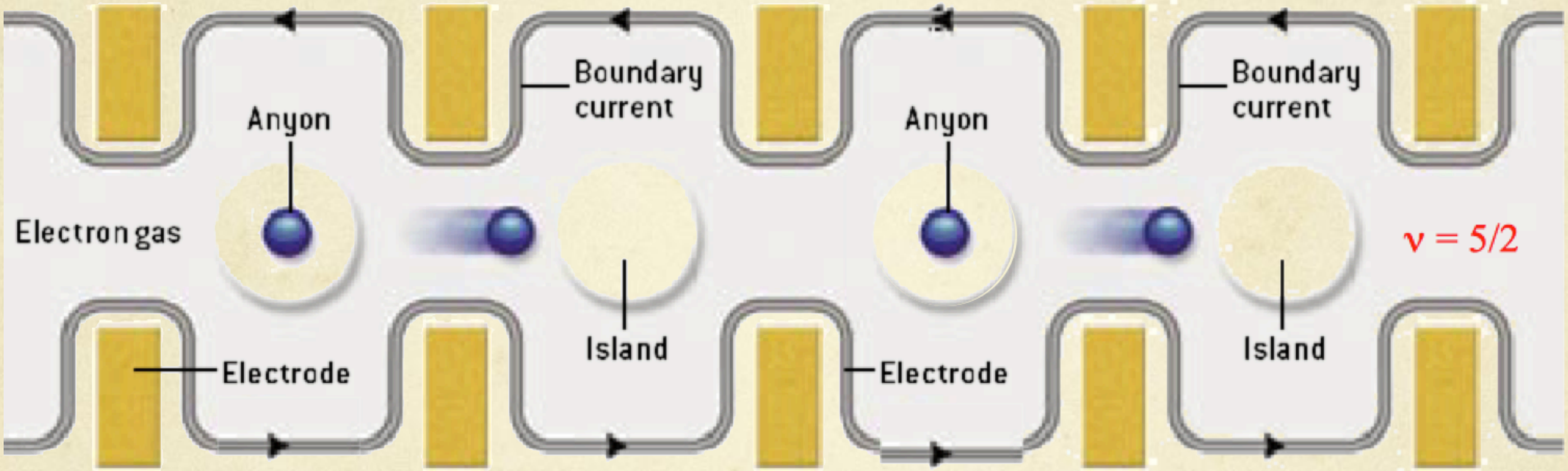


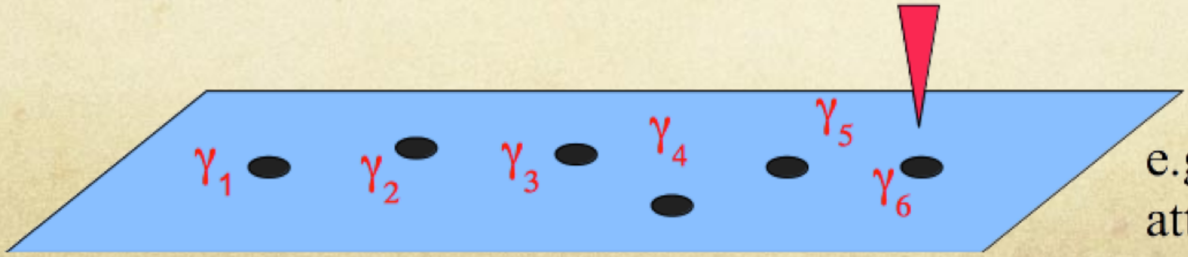
exactly one quasihole added

Switch of the e/4 and e/2 dominated patterns.

Conceptual Design

Das Sarma et al. (05)





e.g., use AFM tips to attract and move anyons

Chiral p-wave Superconductors

- The Moore-Read/Pfaffian wave function is (asymptotically) the same as the BCS wave function with weak complex *p*-wave pairing.

- The effective Bogoliubov-de Gennes Hamiltonian

$$H_{BdG} = \sum_k \left[\xi_k c_k^+ c_k + \frac{1}{2} \left(\Delta_k^* c_{-k} c_k + \Delta_k c_k^+ c_{-k}^+ \right) \right]$$

$$\xi_k = \epsilon_k - \mu , \quad \Delta_k = \hat{\Delta} (k_x - i k_y)$$

Trail wave function $\quad |\Omega\rangle = \prod_k{}' (u_k + v_k c_k^+ c_{-k}^+) |0\rangle , \quad |u_k|^2 + |v_k|^2 = 1$

Weak Coupling

- BdG transformation

$$\alpha_k = u_k c_k - v_k c_{-k}^+, \qquad \alpha_k^+ = u_k^* c_k^+ - v_k^* c_{-k}$$

$$H_{BdG} = \sum_k E_k \alpha_k^+ \alpha_k + const.$$

$$\rightarrow \qquad E_k = \sqrt{\xi_k^2 + |\Delta_k|^2} \qquad\qquad |u_k|^2 = \frac{1}{2}\left(1 + \frac{\xi_k}{E_k}\right)$$

$$|v_k|^2 = \frac{1}{2}\left(1 - \frac{\xi_k}{E_k}\right)$$

Weak pairing phase: $\xi_k \sim -m < 0$ for $k \rightarrow 0$, so asymptotically

$$u_k \propto k_x + i k_y, \qquad v_k = 1$$

Read & Green, PRB 2000

Moore-Read Pfaffian

- Because of Fermi statistics, we can write

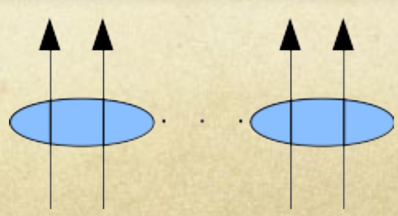
$$|\Omega\rangle = \left(\prod_k |u_k|^{1/2}\right)\exp\left(\frac{1}{2}\sum_k g_k c_k^+ c_{-k}^+\right)|0\rangle, \qquad g_k \propto \frac{1}{k_x + i k_y}$$

- Projected into real space

$$\Psi(z_{1,}\cdots, z_N) = \langle 0| c_{z_1}\cdots c_{z_N}|\Omega\rangle \propto Pf\, g(z_i - z_j) = Pf\,\frac{1}{z_i - z_j}, \qquad g(z)\propto\frac{1}{z}$$

$$Pf\,\frac{1}{z_i-z_j} = A\left(\frac{1}{z_1-z_2}\,\frac{1}{z_3-z_4}\cdots\frac{1}{z_{2N-1}-z_{2N}}\right)$$

$$\Psi_{1/2} = Pf\left(\frac{1}{z_i-z_j}\right)\prod\nolimits_{i<j}(z_i-z_j)^2$$

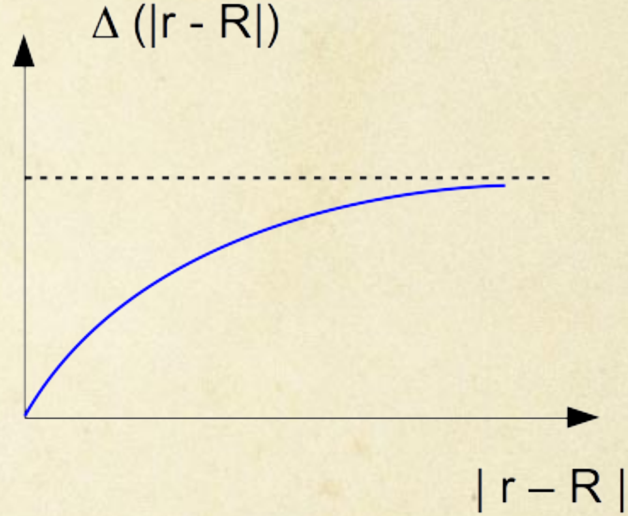


Majorana Zero Modes

- Vortices are introduced by $\Delta(r) = |\Delta(r - R)| e^{i\theta + i\Omega}$
- For p+ip wave pairing, the BdG equation has a single zero-energy solution, localized close to R

$$\Gamma_{E=0} = \gamma = \frac{1}{\sqrt{2}} \left[d\, e^{-i\Omega/2} + d^{+} e^{i\Omega/2} \right]$$

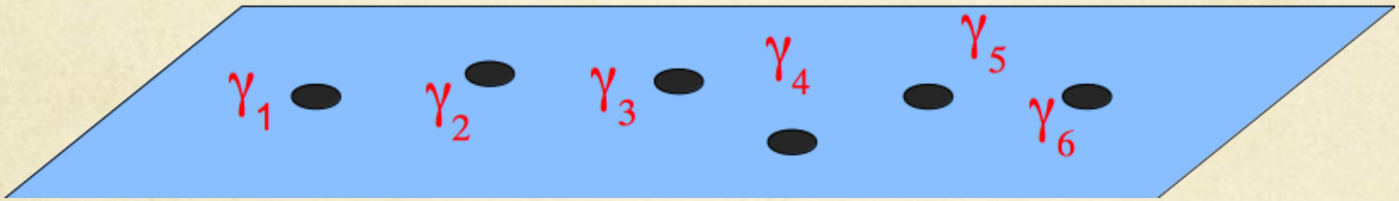
$$\gamma^{+} = \gamma \qquad \gamma^{2} = 1$$



- Ω: Global phase determined by the location of other vortices. And *d* annihilates a particle in the localized vortex-core state.

Ground State Degeneracy

- A system of $2n$ well separated vortices, each carrying one Majorana zero mode, gives rise to the degeneracy of the ground state as 2^n.



$$c_1 = \gamma_1 + i\gamma_2 \qquad |0\rangle,\ c_1^+ c_2^+|0\rangle,\ c_1^+ c_3^+|0\rangle,\ c_2^+ c_3^+|0\rangle$$

$$c_2 = \gamma_3 + i\gamma_4$$

$$c_3 = \gamma_5 + i\gamma_6 \qquad c_1^+|0\rangle,\ c_2^+|0\rangle,\ c_3^+|0\rangle,\ c_1^+ c_2^+ c_3^+|0\rangle$$

New directions: Majoranas bound to defects.

From Dirac to Majorana

- Complex fermions $\{c_i, c_j^+\} = \delta_{ij}$
- Linear combinations of particles and holes

$$\gamma_{2i-1} = c_i + c_j^+, \quad \gamma_{2i} = c_i - c_j^+$$

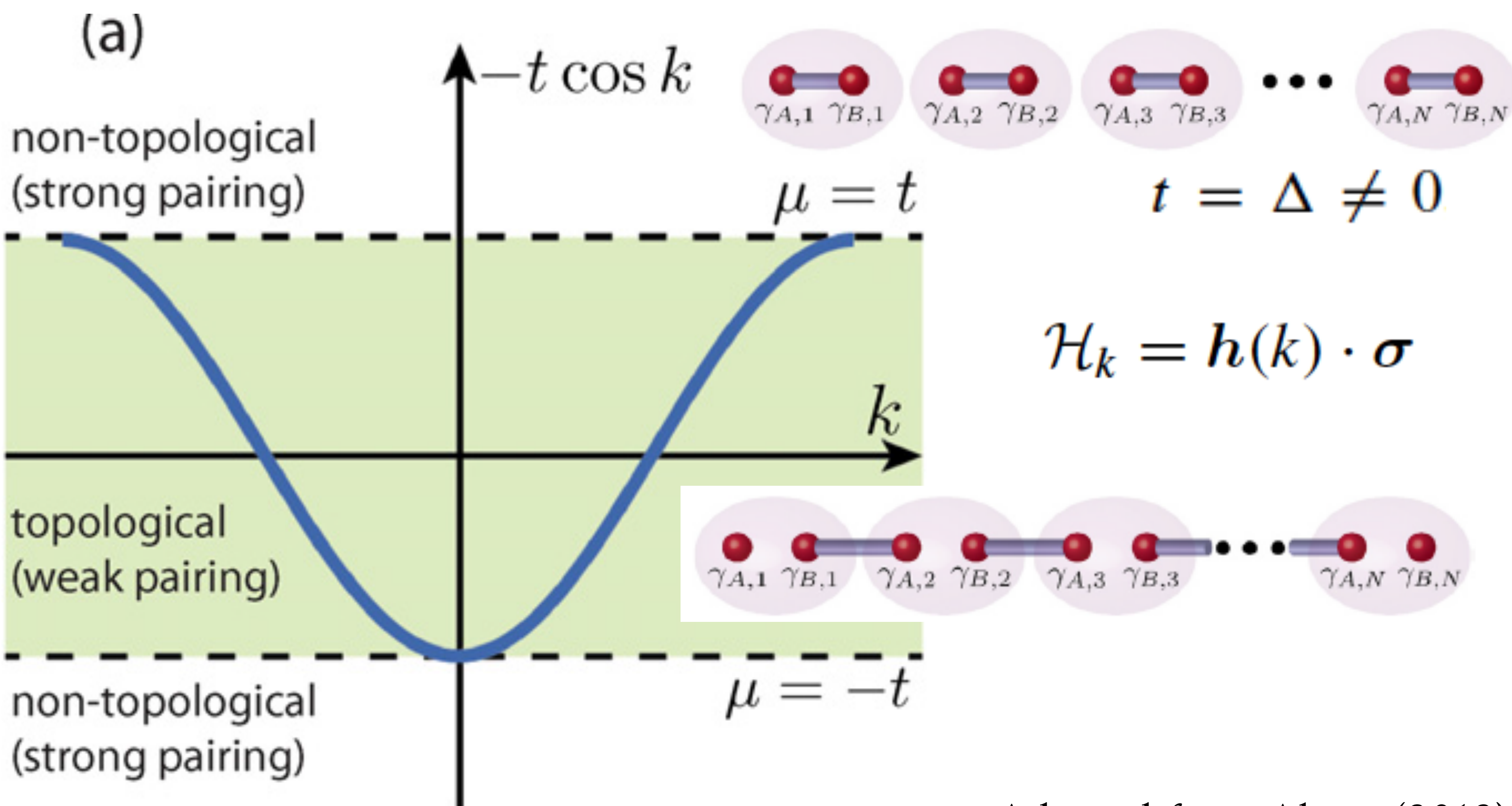
$$\{\gamma_i, \gamma_j\} = 2\delta_{ij}$$

- Problem: The two Majoranas appear at the same site, hence trivial. We need spinless fermions, superconducting pairing (hence mixing particles and holes), and a twist.

See, e.g., J Alicea, Rep. Prog. Phys. **75**, 076501 (2012)

Kitaev's Toy Model

$$H = -\mu \sum_x c_x^+ c_x - \frac{1}{2} \sum_x \left(t c_x^+ c_{x+1} + \Delta c_x c_{x+1} + h.c. \right)$$



Adapted from Alicea (2012)

TFIM vs p-wave SC

From Lecture 2:

$$\boxed{H_S = -J\sum_{j=1}^{N-1}\sigma_j^x\sigma_{j+1}^x - h\sum_{j=1}^{N}\sigma_j^z}$$

$$a_j = \left(\prod_{k=1}^{j-1}\sigma_k^z\right)\sigma_j^+ \qquad a_j^+ = \left(\prod_{k=1}^{j-1}\sigma_k^z\right)\sigma_j^- \qquad \sigma_j^z = (-1)^{a_j^+ a_j}$$

$$\Rightarrow \quad \sigma_j^x\sigma_{j+1}^x = -\left(a_j - a_j^+\right)\left(a_{j+1} + a_{j+1}^+\right)$$

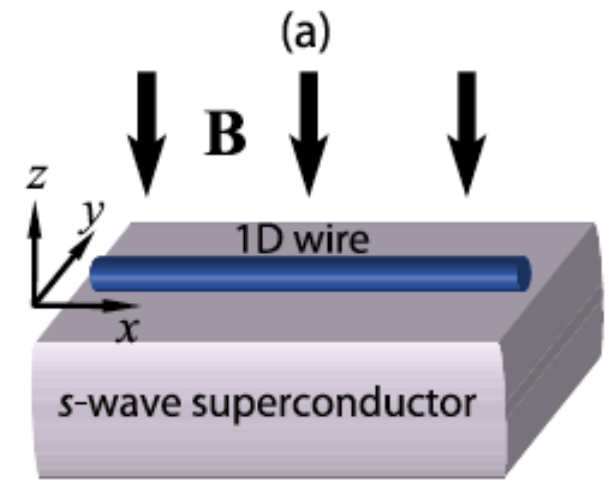
Jordan-Wigner Transformation

$$H = J\sum_{j=1}^{N-1}\left(a_j - a_j^+\right)\left(a_{j+1} + a_{j+1}^+\right) - 2h\sum_{j=1}^{N}\left(a_j^+ a_j - \frac{1}{2}\right)$$

Previous slide:

$$\boxed{H = -\mu\sum_x c_x^+ c_x - \frac{1}{2}\sum_x\left(tc_x^+ c_{x+1} + \Delta c_x c_{x+1} + h.c.\right)}$$

1D Quantum Wire

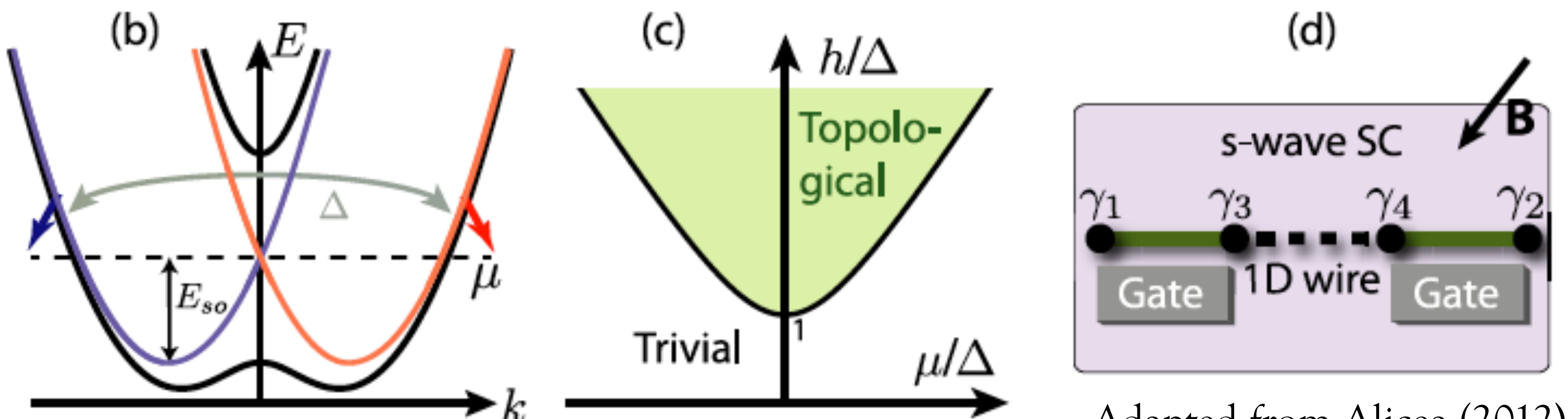


$$H = H_{\rm wire} + H_\Delta,$$

$$H_{\rm wire} = \int dx\,\psi^\dagger\left(-\frac{\partial_x^2}{2m} - \mu - i\alpha\sigma^y\partial_x + h\sigma^z\right)\psi,$$

$$H_\Delta = \int dx\,\Delta(\psi_\uparrow\psi_\downarrow + \mathrm{H.c.}).$$

$$h > \sqrt{\Delta^2 + \mu^2}$$

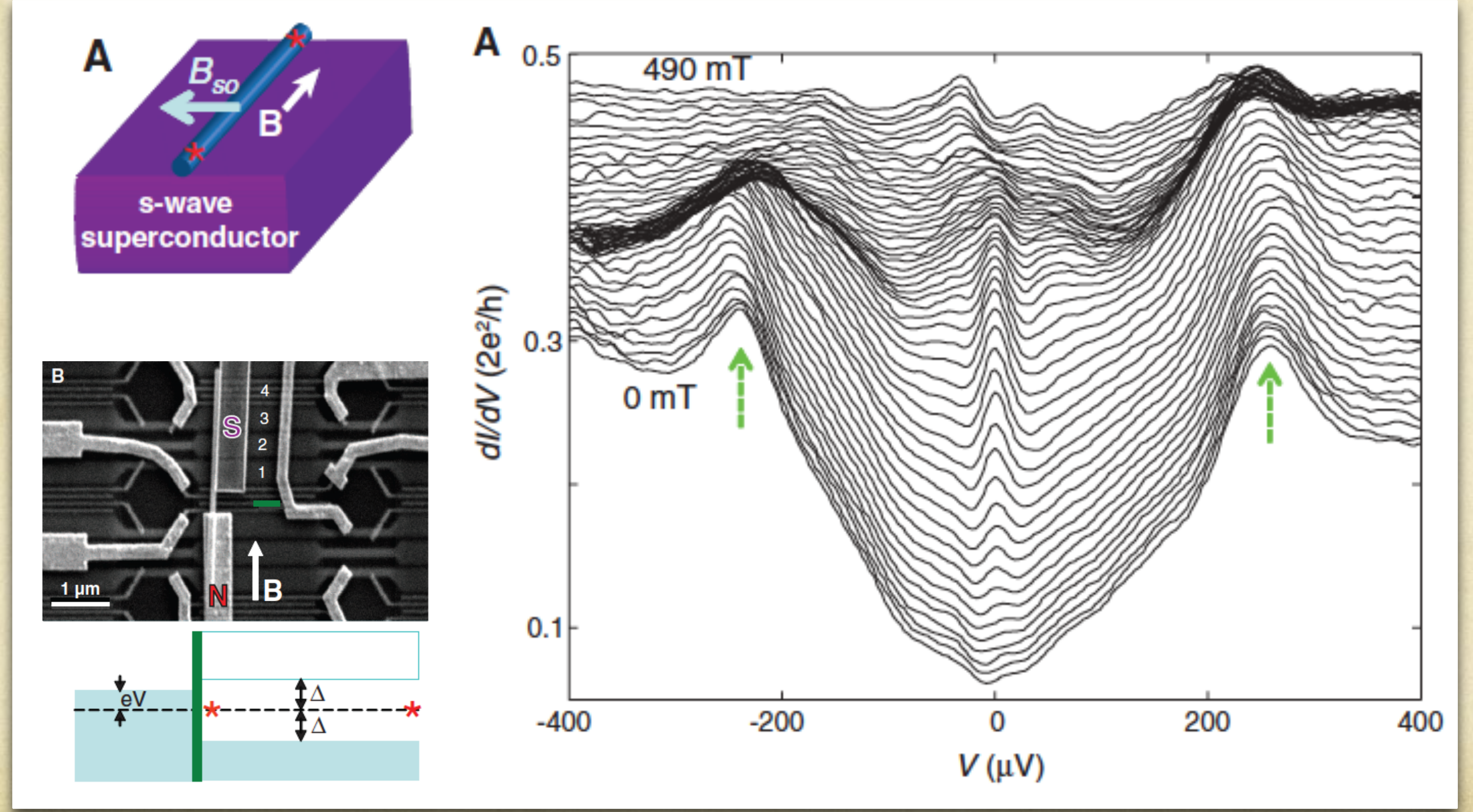


Adapted from Alicea (2012)

Experimental Progress

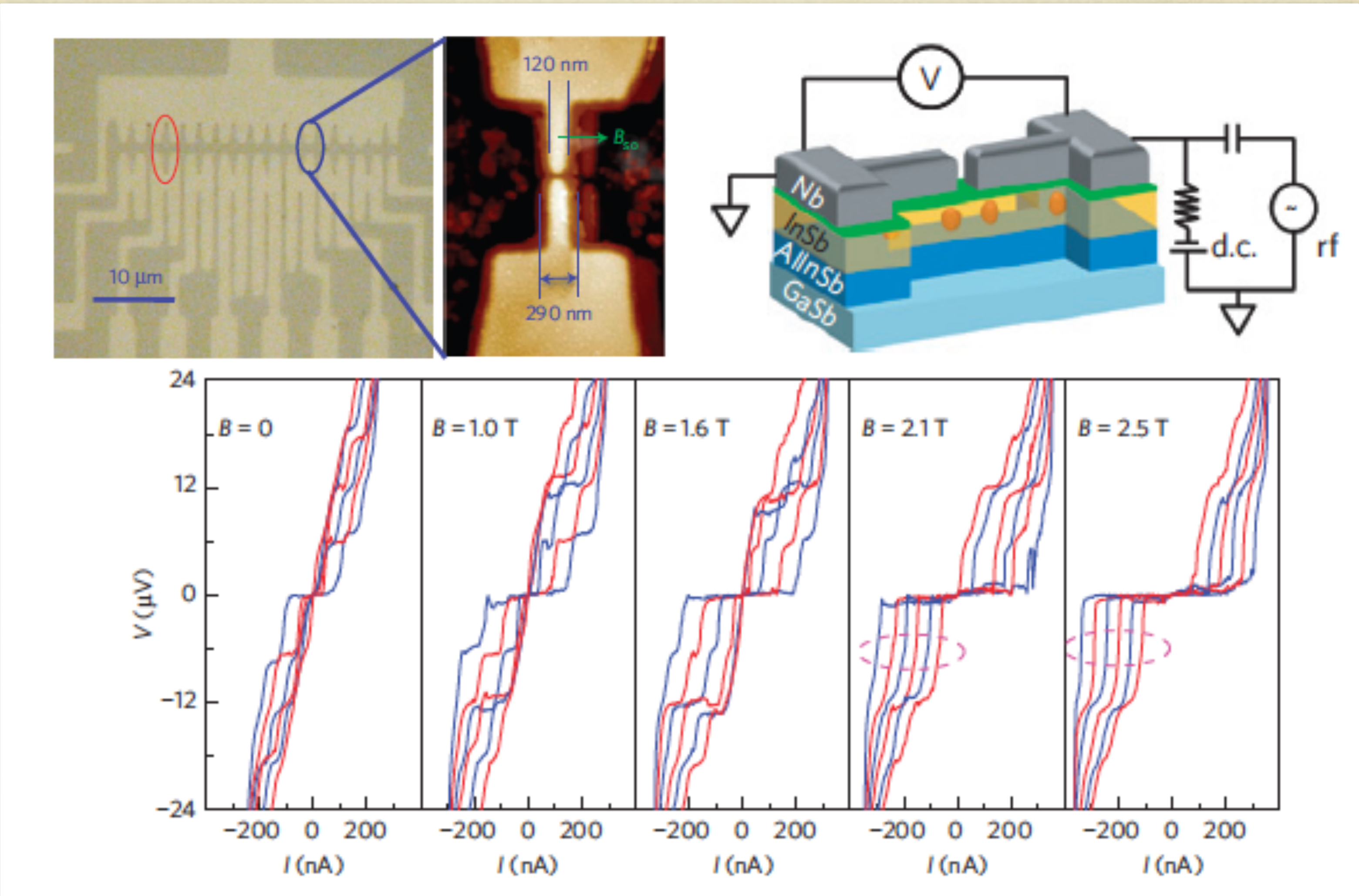
- Identified possible signatures of Majorana modes in nanowires with proximity-induced superconductivity, with small amounts of mode splitting potentially explained by hybridization of Majorana modes.
  - Anomalous zero-bias conductance peak (Mourik et al., 2012; Deng et al., 2012; Das et al., 2012; Churchill et al., 2013)
  - Fractional a.c. Josephson effect (Rokhinson et al., 2012)
  - Exponential splitting of zero modes (Albrecht et al., 2016)

Zero-Bias Anomaly



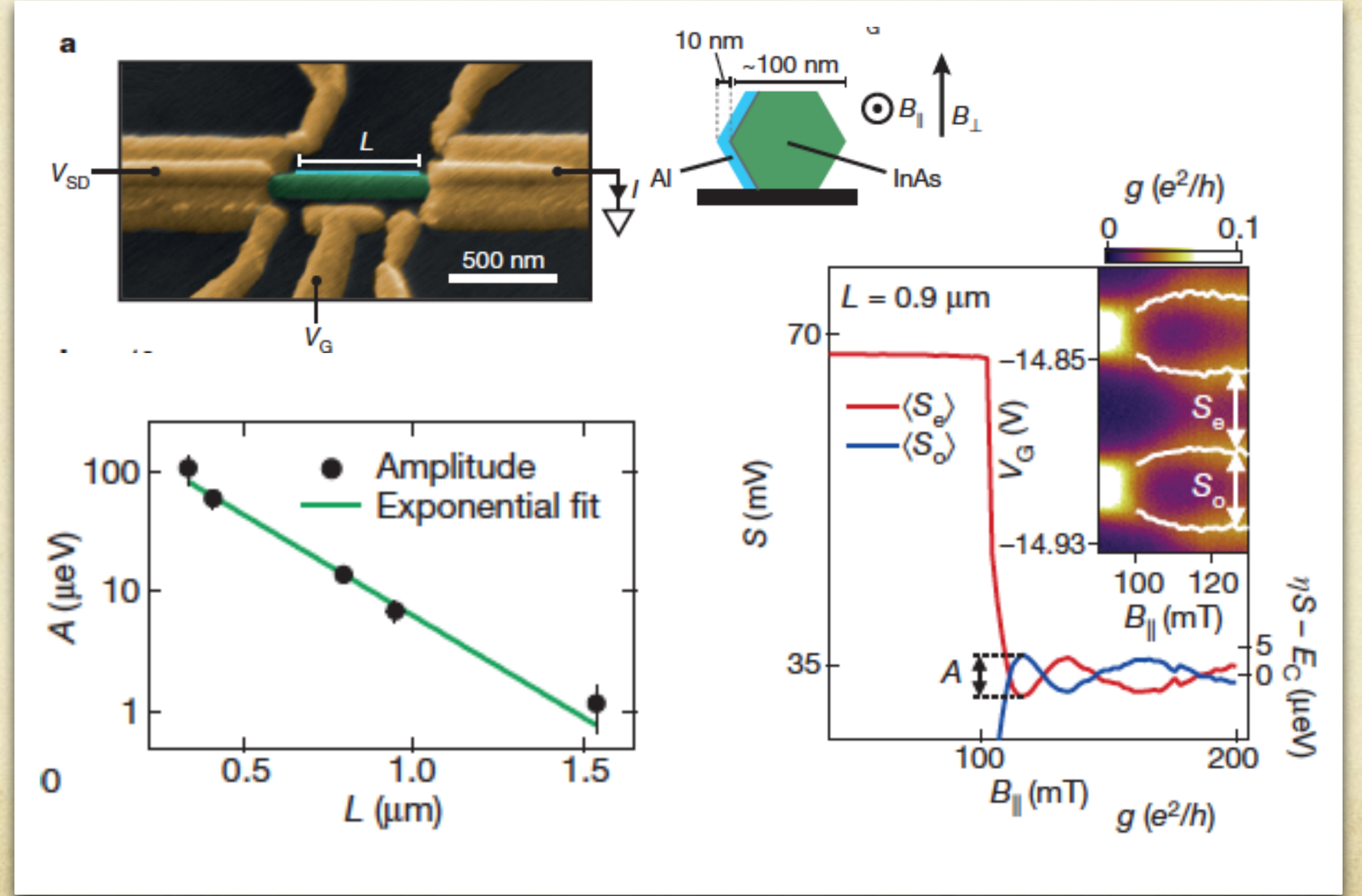
Mourik et al., Science 336, 1003 (2012)

Fractional ac Josephson Effect



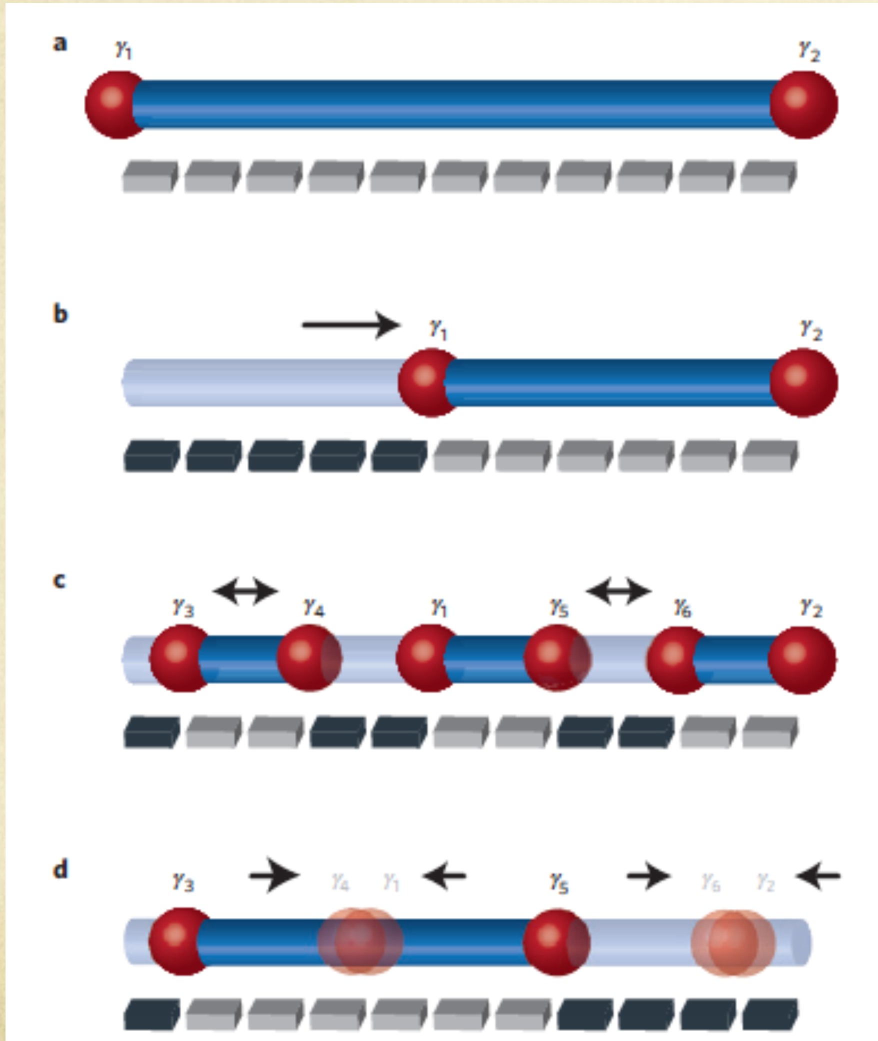
Rokhinson et al., Nature Phys. 8, 795 (2012)

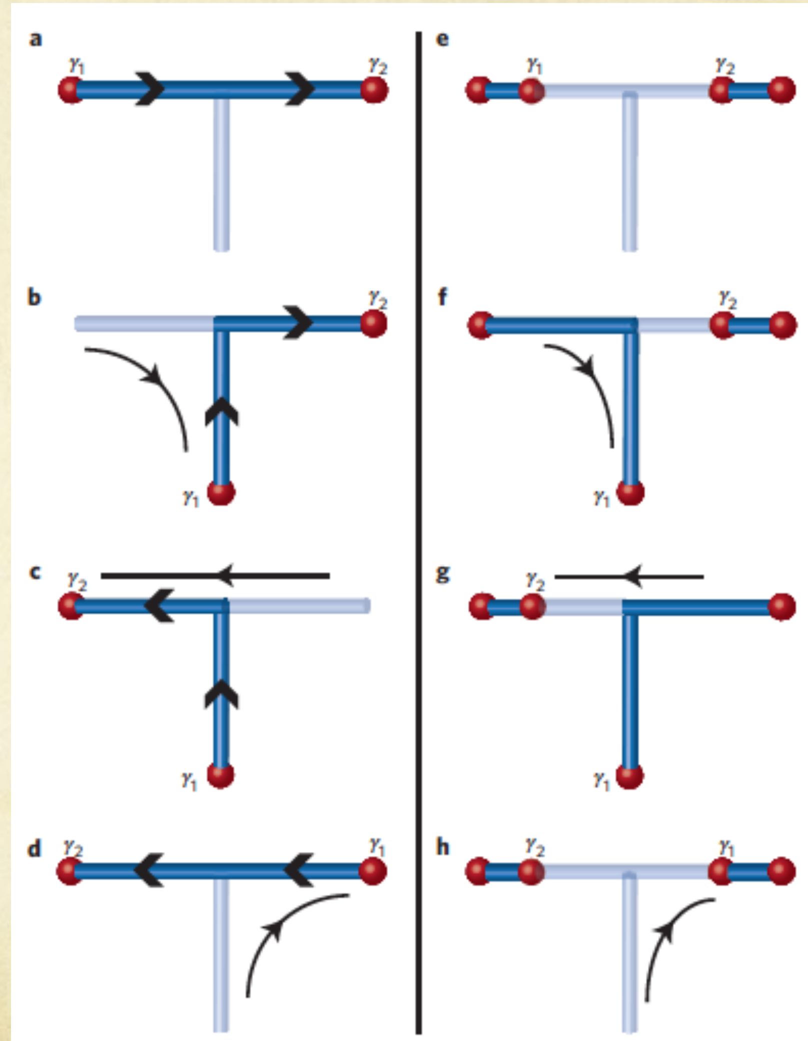
Exponential Protection



Albrecht et al., Nature 531, 206 (2016)

Initialization, Fusion and Braiding





Alicea, Oreg, Rafael, von Oppen, & Fisher, Nature Phys. **7**, 412 (2011)

Conclusion

- The central issue of topological quantum computation is to search for topological systems that can achieve quantum error correction on the hardware level.
- Topological phases can be classified by discrete symmetries and/or interaction.
- Leading candidate systems for topological quantum computation include FQH liquids and quasi-1D quantum wires.

Quantum Manifesto (2016): EU to launch €1 billion quantum technology megaproject in 2018.