Matrix Product States and Their Applications

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Motivation

- Understanding many-body quantum matter, in particular, fractional quantum Hall systems with Abelian or non-Abelian statistics, from quantum information point of view.
- Current (numerical) machineries
  - Explicit expansion (e.g., by Mathematica)
  - Exact diagonalization
  - Jack polynomial (solving partial differential equation)
  - Density-matrix renormalization group
  - Matrix product states
  - Variational quantum Monte Carlo simulation
  - iDMRG
- In addition, a good exercise to learn conformal field theories

Outline

- AKLT state and Introduction to MPS
  - Half-integer and integer spin chains
  - AKLT states
  - Matrix product states
  - Quantum entanglement

- iTEBD and its applications
- MPS of the Laughlin state

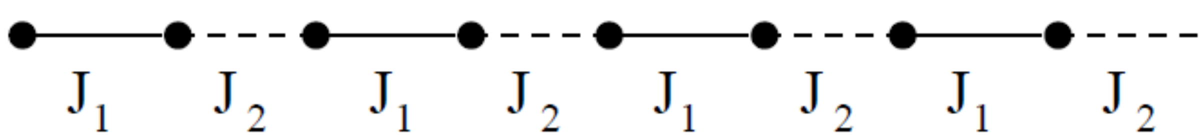
- Summary

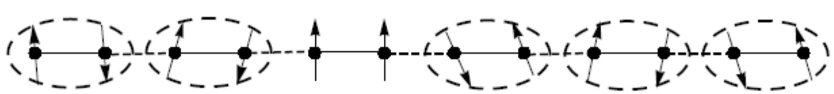
One-Dimensional Spin Chains I

- Heisenberg AFM chain:

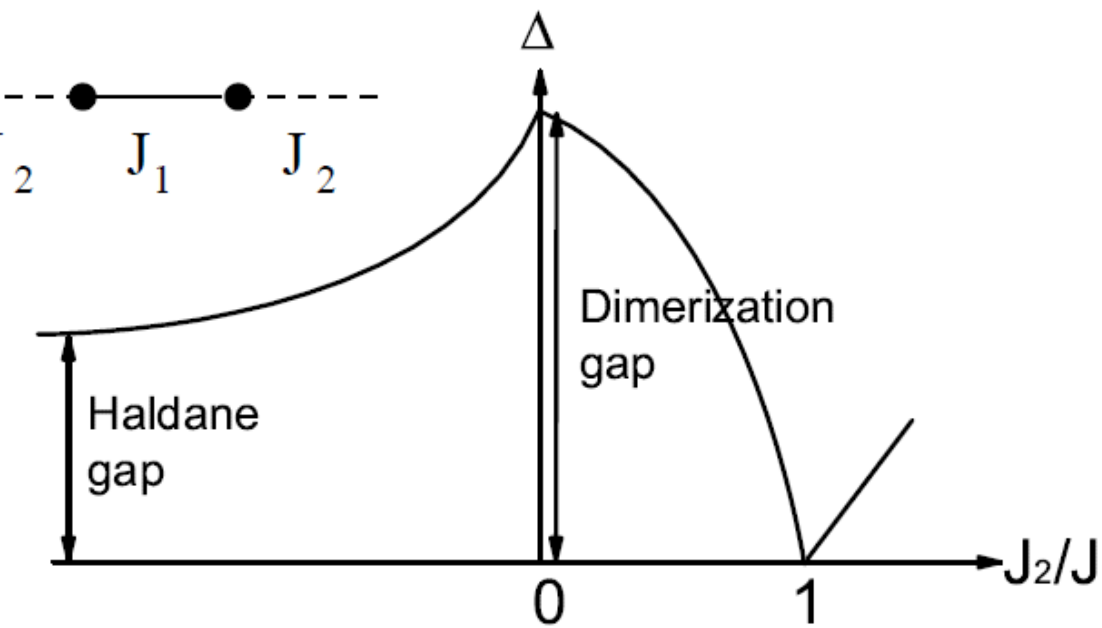
$$H = J \sum_i S_i \cdot S_{i+1}$$

- S = 1/2, solved by Bethe ansatz
  - Non-degenerate ground state
  - Gapless excitations (Lieb-Schultz-Mattis theorem)
  - Power law decay of spin-spin correlations – quantum critical





A localized magnon carrying spin-1 of the J1−J2 model

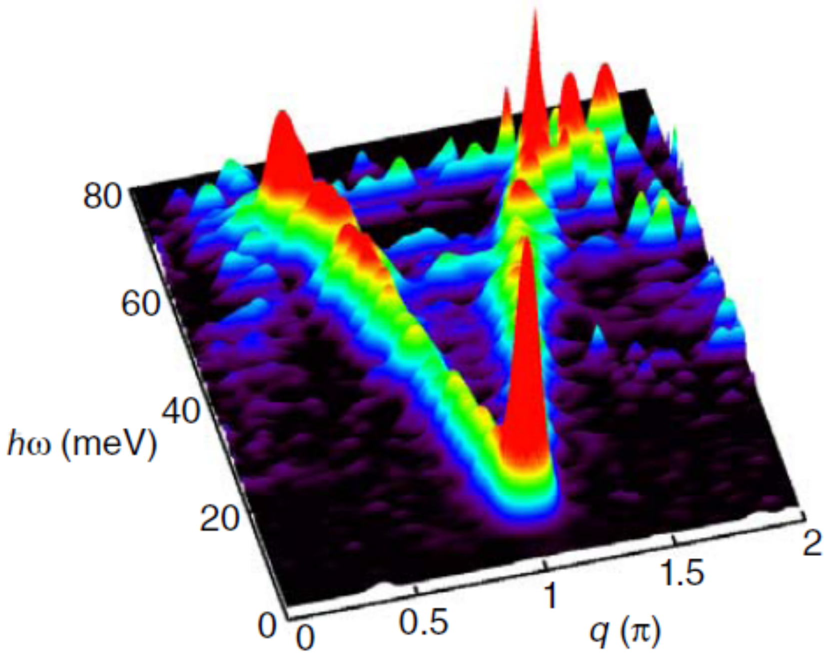


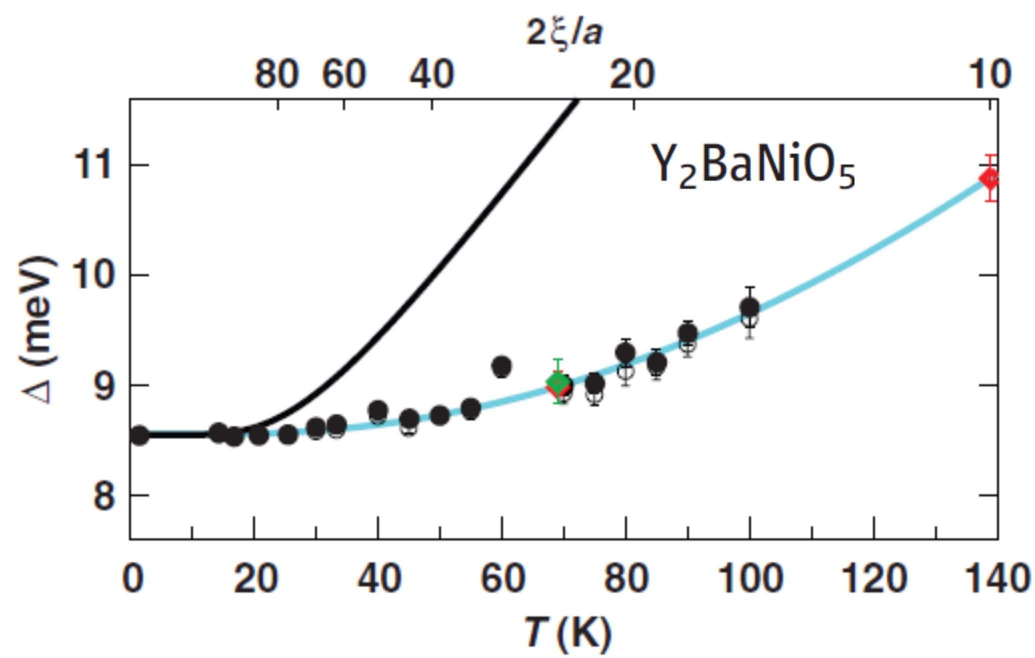
One-Dimensional Spin Chains II

- Heisenberg AFM chain:

$$H = J\sum_i S_i \cdot S_{i+1}$$

- S = 1, Haldane conjecture
  - Gapped excitations
  - Exponentially decaying spin-spin correlation



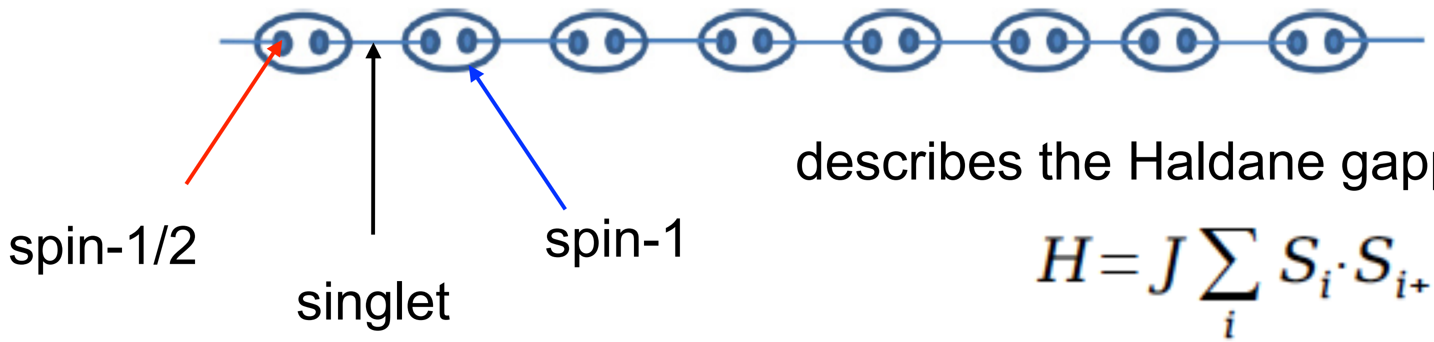


Xu et al., Science 317, 1049 (2007)

One-Dimensional Spin Chains III

I. Affleck, T. Kennedy, E. H. Lieb, and H. Tasaki, Commun. Math. Phys. 115, 477 (1988)

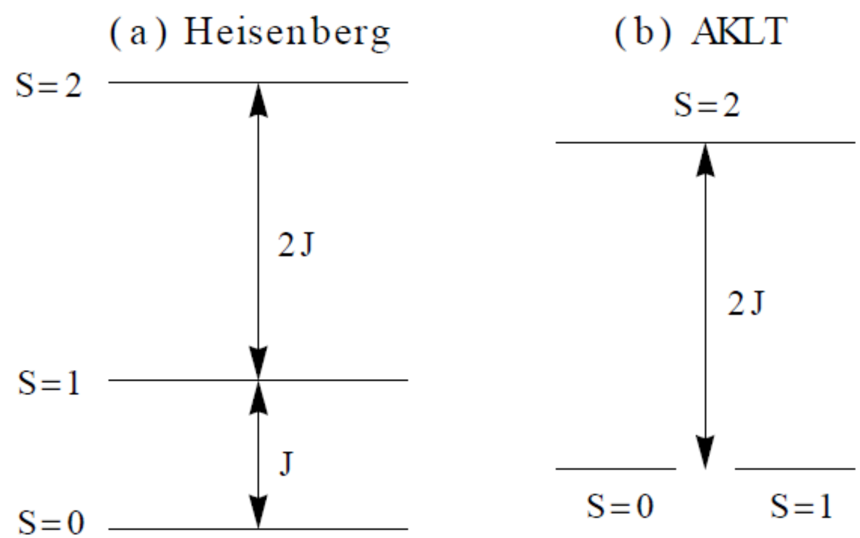
Spin-1: $$H=\frac{1}{2}\sum_i\left[\vec{S}_i\cdot\vec{S}_{i+1}+\frac{1}{3}\left(\vec{S}_i\cdot\vec{S}_{i+1}\right)^2+\frac{2}{3}\right]=\sum_i P_2(i,i+1)$$



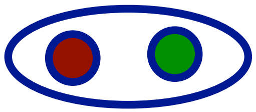
describes the Haldane gapped phase

$$H=J\sum_i S_i\cdot S_{i+1}$$

FIG: Spectra of two spin-1's coupled through (a) AFM Heisenberg and (b) AKLT couplings. The singlet (S = 0) and triplet (S = 1) states are degenerate ground states for the AKLT coupling.



The Spin-1 AKLT Chain: Ground State



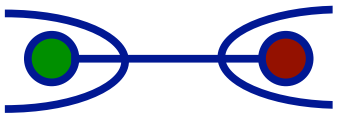
Spin-1 replaced by a pair of symmetrized spin-1/2

$$P = |+\rangle\langle\uparrow\uparrow| + |0\rangle\frac{\langle\uparrow\downarrow|+\langle\downarrow\uparrow|}{\sqrt{2}} + |-\rangle\langle\downarrow\downarrow|$$

$$P^{\sigma_i} = \sum_{\sigma_i}\sum_{a_i,b_i} P^{\sigma_i}_{a_i b_i}\,|\sigma_i\rangle\langle a_i b_i|$$

$$P^{+} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad P^{0} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad P^{-} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

The Spin-1 AKLT Chain: Ground State



Adjacent pair on neighboring sites form a singlet

$$\frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}$$

$$= \sum_{b_i, a_{i+1}} S_{b_i a_{i+1}} |b_i\rangle |a_{i+1}\rangle \qquad S = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$|\Psi_S\rangle = \sum_{\{a_i\}} \sum_{\{b_i\}} S_{b_1 a_2} S_{b_2 a_3} \cdots S_{b_L a_1} |\{a_i\}, \{b_i\}\rangle$$

periodic boundary conditions

The Spin-1 AKLT Chain: Ground State

$$\left|\tilde{\Psi}\right\rangle = \sum_{\{\sigma_i\}}\sum_{\{a_i\}}\sum_{\{b_i\}} P^{\sigma_1}_{a_1 b_1} P^{\sigma_2}_{a_2 b_2} \cdots P^{\sigma_L}_{a_L b_L} \left|\{\sigma_i\}\right\rangle\left\langle \{a_i\},\{b_i\}\right|\Psi_S\rangle$$

$$= \sum_{\{\sigma_i\}}\sum_{\{a_i\}}\sum_{\{b_i\}} P^{\sigma_1}_{a_1 b_1} S_{b_1 a_2} P^{\sigma_2}_{a_2 b_2} S_{b_2 a_3} \cdots P^{\sigma_L}_{a_L b_L} S_{b_L a_1} \left|\{\sigma_i\}\right\rangle$$

$$= \sum_{\{\sigma_i\}} Tr\left(A^{\sigma_1} A^{\sigma_2} \cdots A^{\sigma_L}\right)\left|\{\sigma_i\}\right\rangle \qquad \boxed{A^{\sigma}_{aa'} = P^{\sigma}_{ab} S_{ba'}}$$

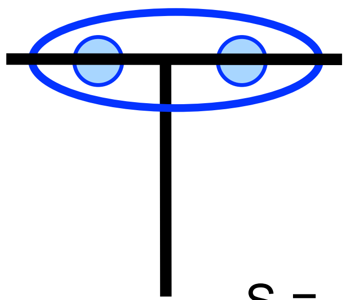
$$A^{[-1]} = \begin{pmatrix} 0 & 0 \\ -\sqrt{\frac{2}{3}} & 0 \end{pmatrix}, \quad A^{[0]} = \begin{pmatrix} -\frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{3}} \end{pmatrix}, \quad A^{[+1]} = \begin{pmatrix} 0 & \sqrt{\frac{2}{3}} \\ 0 & 0 \end{pmatrix}$$

Up to a normalization factor (Can you see why?)

Matrix-Product State

auxiliary
space

edge
spin-1/2



S = -1, 0, 1

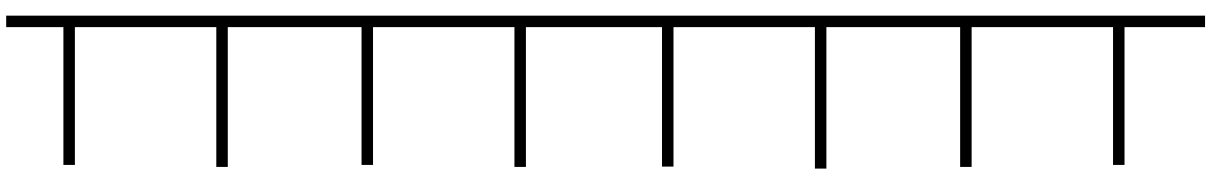
physical degree of
freedom (spin-1)

$$A^{[-1]} = \begin{pmatrix} 0 & 0 \\ -\sqrt{\frac{2}{3}} & 0 \end{pmatrix}, \quad A^{[0]} = \begin{pmatrix} -\frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{3}} \end{pmatrix},$$

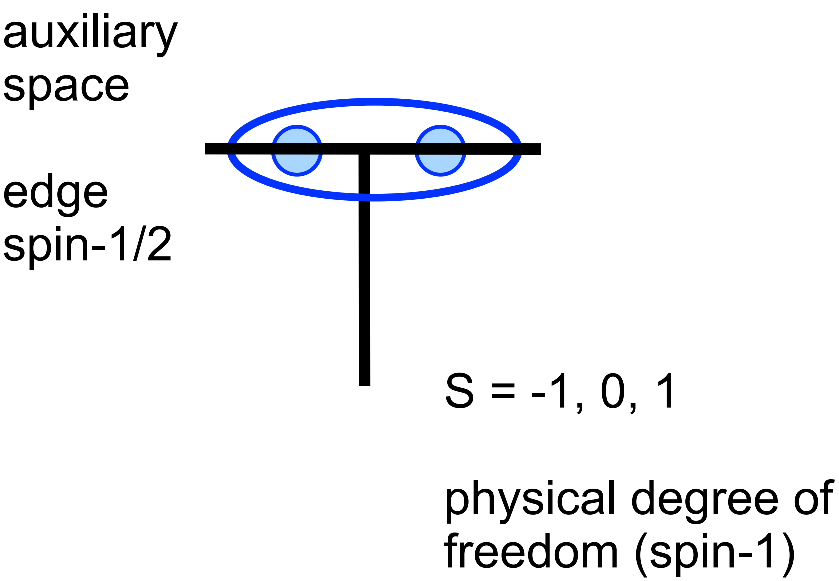
$$A^{[+1]} = \begin{pmatrix} 0 & \sqrt{\frac{2}{3}} \\ 0 & 0 \end{pmatrix}$$

Matrix-product
representation

$$|\Psi\rangle = \sum_{\{s_i\}} \mathrm{Tr}(\mathbf{A}^{[s_1]}\mathbf{A}^{[s_2]}...\mathbf{A}^{[s_L]})|s_1, s_2, \ldots s_L\rangle$$



Realization of the AKLT State in Python



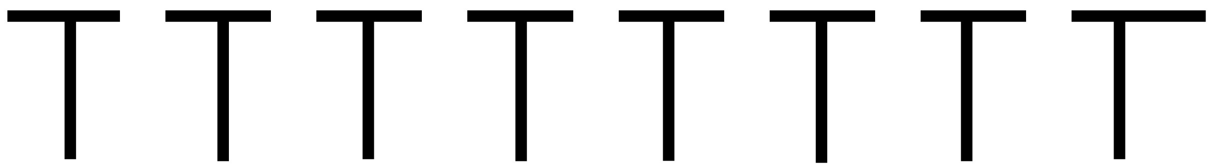

```
import numpy as np
import math

# First define the AKLT tensor
M = np.zeros([3,2,2])
M[0,1,0] = -math.sqrt(2.0/3.0)
M[1,0,0] = -math.sqrt(1.0/3.0)
M[1,1,1] = math.sqrt(1.0/3.0)
M[2,0,1] = math.sqrt(2.0/3.0)
M = np.transpose(M,(1,0,2))

print (M)
```

$$A^{[-1]} = \begin{pmatrix} 0 & 0 \\ -\sqrt{\frac{2}{3}} & 0 \end{pmatrix}, \quad A^{[0]} = \begin{pmatrix} -\frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{3}} \end{pmatrix}, \quad A^{[+1]} = \begin{pmatrix} 0 & \sqrt{\frac{2}{3}} \\ 0 & 0 \end{pmatrix}$$

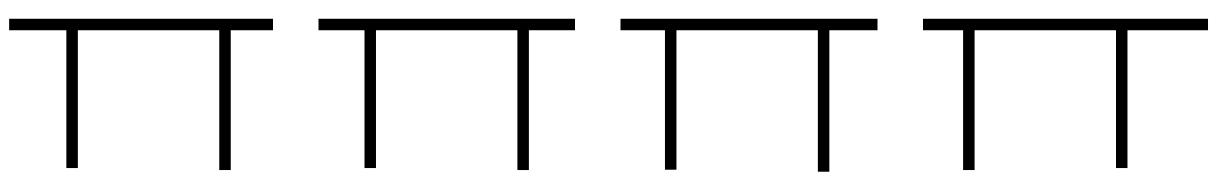
Realization of the AKLT State in Python

Matrix-product representation

$$|\Psi\rangle = \sum_{\{s_i\}} \mathrm{Tr}(\mathbf{A}^{[s_1]}\mathbf{A}^{[s_2]}...\mathbf{A}^{[s_L]})|s_1, s_2, \ldots s_L\rangle$$

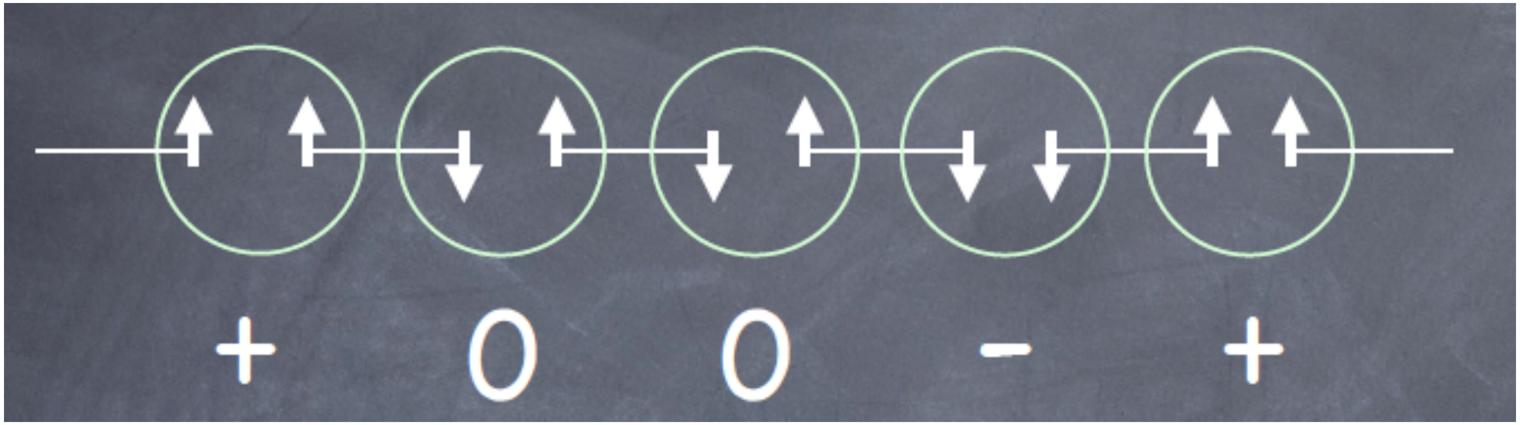


```
# 2N-leg tensor
A = np.tensordot(M,M,axes=(2,0))
B = A
for i in range(1,N):
   B = np.tensordot(B,A,axes=(2*i+1,1))
```



Hidden Antiferromagnetic String Order

In Sz basis, + and – alternate, with any number of 0’s in between.



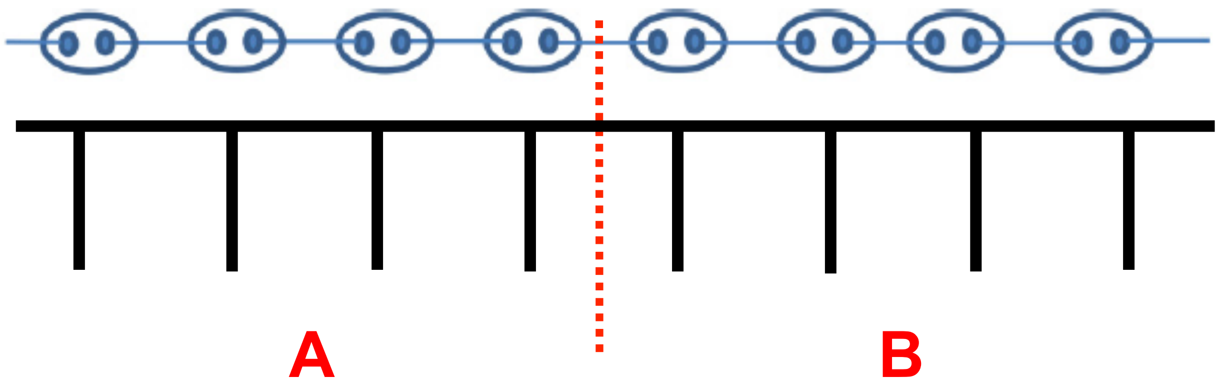
+0000−	000+0−	+−00+−	0+−+−0	+0−000	−+−+00	00−0+0	−0+0−+	0−+−0+
0+000−	+00−+−	0000+−	+−0+−0	0+−000	0−0+00	−+−0+0	00−+−+	−0+−0+
+−+00−	0+0−+−	+000−0	000+−0	+−0000	−00+00	0−00+0	−+−+−+	00−00+
00+00−	+−+−+−	0+00−0	+00−00	−+0000	000−+0	−000+0	0−0+−+	−+−00+
+0−+0−	00+−+−	+−+0−0	0+0−00	0−+000	−+0−+0	0000−+	−00+−+	0−000+
0+−+0−	+0−0+−	00+0−0	+−+−00	−0+000	0−+−+0	−+00−+	000−0+	−0000+
+−0+0−	0+−0+−	+0−+−0	00+−00	00−+00	−0+−+0	0−+0−+	−+0−0+	

$$O^{\alpha}_{string} = \lim_{|j-k|\to\infty} \langle \psi_0 | S^{\alpha}_j \exp\left[i\pi \sum_{l=j+1}^{k-1} S^{\alpha}_l \right] S^{\alpha}_k | \psi_0 \rangle = \frac{4}{9}$$

M. den Nijs and K. Rommelse,
PRB **40**, 4609 (1989).

Bipartition and Entanglement Spectrum

$|\Psi\rangle =$



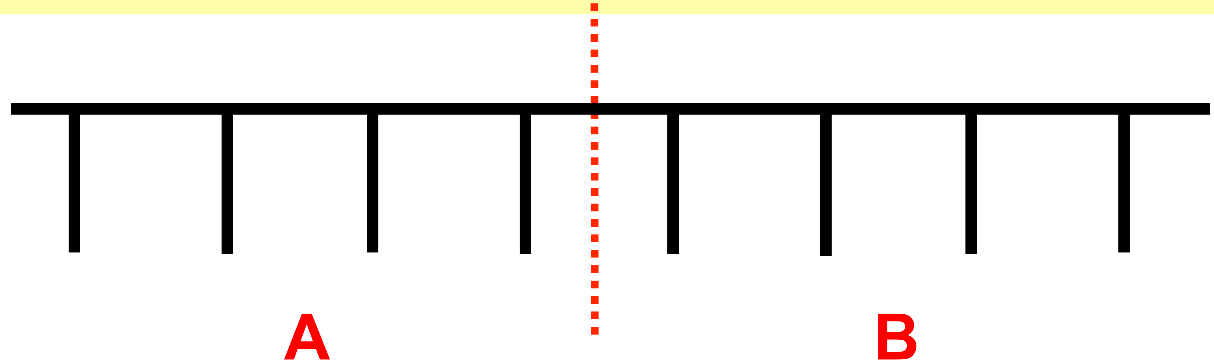
Singular value decomposition

$$|\Psi\rangle = \sum_i e^{-\xi_i/2} \left|\psi_i^A\right\rangle \left|\psi_i^B\right\rangle$$

```
dim = 3**N
E = np.reshape(B,(2*dim,2*dim))
X,Y,Z = np.linalg.svd(E)
S=Y**2/sum(Y**2)
print (S)
```

Bipartition and Entanglement Spectrum

$|\Psi\rangle =$

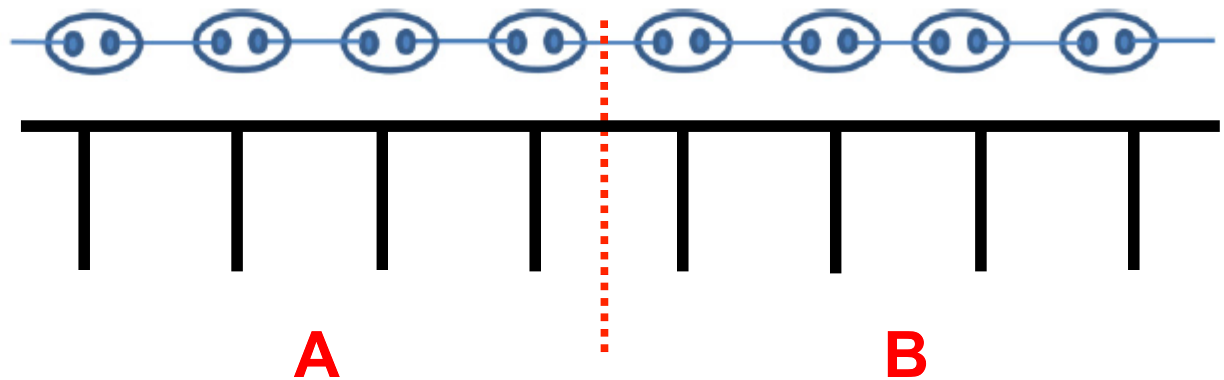



```
[ 5.00000000e-01   5.00000000e-01   6.14365326e-33   6.14365326e-33
  6.14365326e-33   6.14365326e-33   6.14365326e-33   6.14365326e-33
  6.14365326e-33   6.14365326e-33   6.14365326e-33   6.14365326e-33
  6.14365326e-33   6.14365326e-33   6.14365326e-33   6.14365326e-33
  6.14365326e-33   6.14365326e-33   6.14365326e-33   6.14365326e-33
  6.14365326e-33   6.14365326e-33   6.14365326e-33   6.14365326e-33
  6.14365326e-33   6.14365326e-33   6.14365326e-33   6.14365326e-33
  6.14365326e-33   6.14365326e-33   6.14365326e-33   6.14365326e-33
  6.14365326e-33   6.14365326e-33   6.14365326e-33   6.14365326e-33
  6.14365326e-33   6.14365326e-33   6.14365326e-33   6.14365326e-33
  6.14365326e-33   6.14365326e-33   6.14365326e-33   6.14365326e-33
  6.14365326e-33   6.14365326e-33   6.14365326e-33   6.14365326e-33
  6.14365326e-33   6.14365326e-33   6.14365326e-33   6.14365326e-33
  6.12439128e-33   4.55085427e-34]
```

Eigenvalues of ρ_A: $\exp(-\xi_i) \rightarrow$ entanglement spectrum: ξ_i

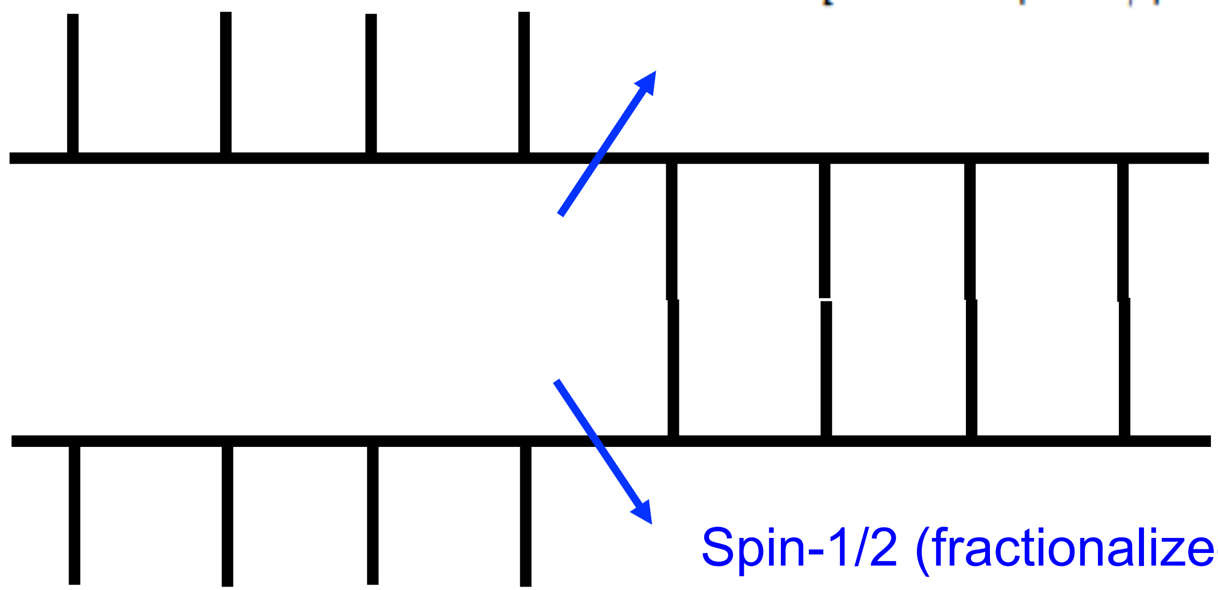
Bipartition and Entanglement Spectrum

$|\Psi\rangle =$



Singular value decomposition $\quad |\Psi\rangle = \sum_i e^{-\xi_i/2} \left|\psi_i^A\right\rangle \left|\psi_i^B\right\rangle$

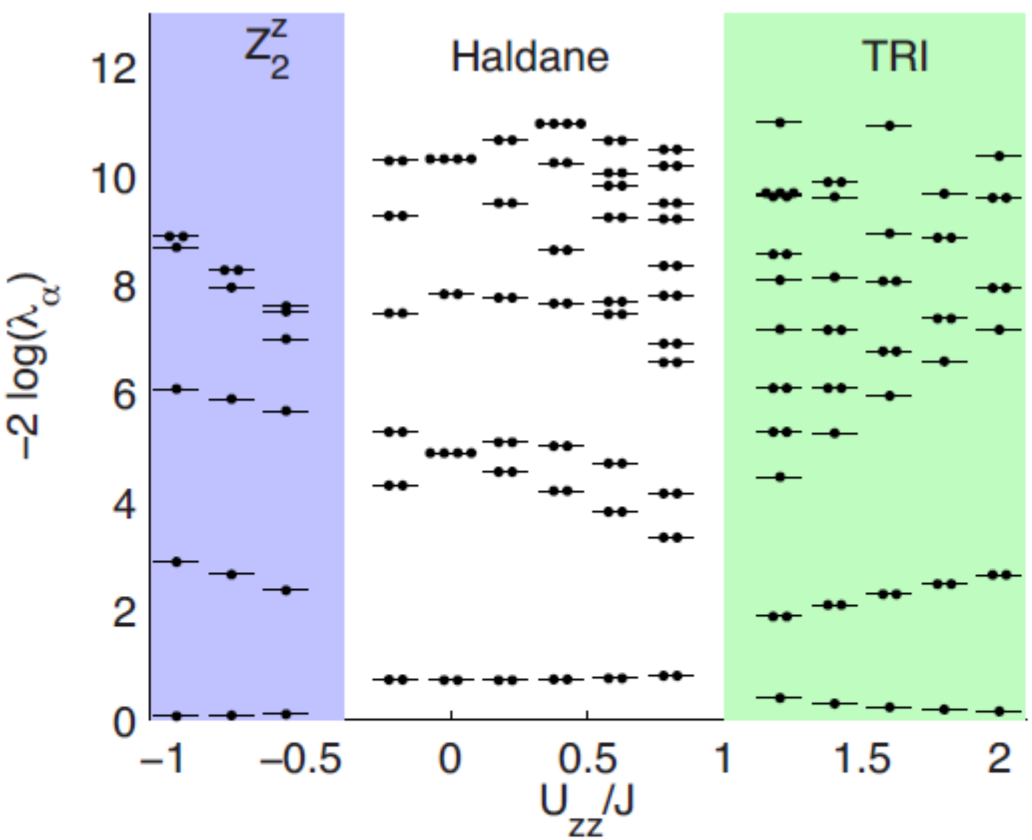
$\rho_A = Tr_B(|\Psi\rangle\langle\Psi|) =$



Eigenvalues of ρ_A: $\exp(-\xi_i)$ $\rightarrow$ entanglement spectrum: ξ_i

Symmetry Protected Topological Phase

Pollmann et al., PRB (2010)



The Haldane phase of S =1 chains is reflected by a double degeneracy of the entire entanglement spectrum.

$$H_0 = J\sum_j \vec{S}_j \cdot \vec{S}_{j+1} + B_x \sum_j S_j^x + U_{zz}\sum_j (S_j^z)^2$$

The degeneracy, or the Haldane phase is protected by any of the following symmetries: time-reversal symmetry, dihedral symmetry D_2, or spatial inversion symmetry.

Outline

- AKLT state and Introduction to MPS
- iTEBD and its applications
  - Transverse-field Ising model
  - Time evolution
  - Decomposing an MPS
  - The iTEBD algorithm
  - Jordan-Wigner transformation
- MPS of the Laughlin state
- Summary

Transverse Field Ising Model

See, e.g, Quantum Phase Transitions by Subir Sachdev.

$$H_S = -J\sum_{j=1}^{N-1}\sigma_j^x\sigma_{j+1}^x - h\sum_{j=1}^{N}\sigma_j^z$$

Large J >> h, $\rightarrow \;\rightarrow \;\rightarrow \;\rightarrow \;\rightarrow \;\rightarrow \;\rightarrow$

or, $\leftarrow \;\leftarrow \;\leftarrow \;\leftarrow \;\leftarrow \;\leftarrow \;\leftarrow$

quantum phase transition

Small J << h, $\uparrow \;\uparrow \;\uparrow \;\uparrow \;\uparrow \;\uparrow \;\uparrow$

Z_2 symmetry by a global flip in the σ_x basis $\quad P_S = \prod_{j=1}^{N}\sigma_j^z$

Introduction to Time Evolution

- Interested in calculating the time evolution of a state
  - Imaginary-time evolution

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle \qquad U(\tau) = \exp(-H\tau)$$

$$|\psi_G\rangle = \lim_{\tau\to\infty} e^{-\tau H}|\psi_0\rangle$$

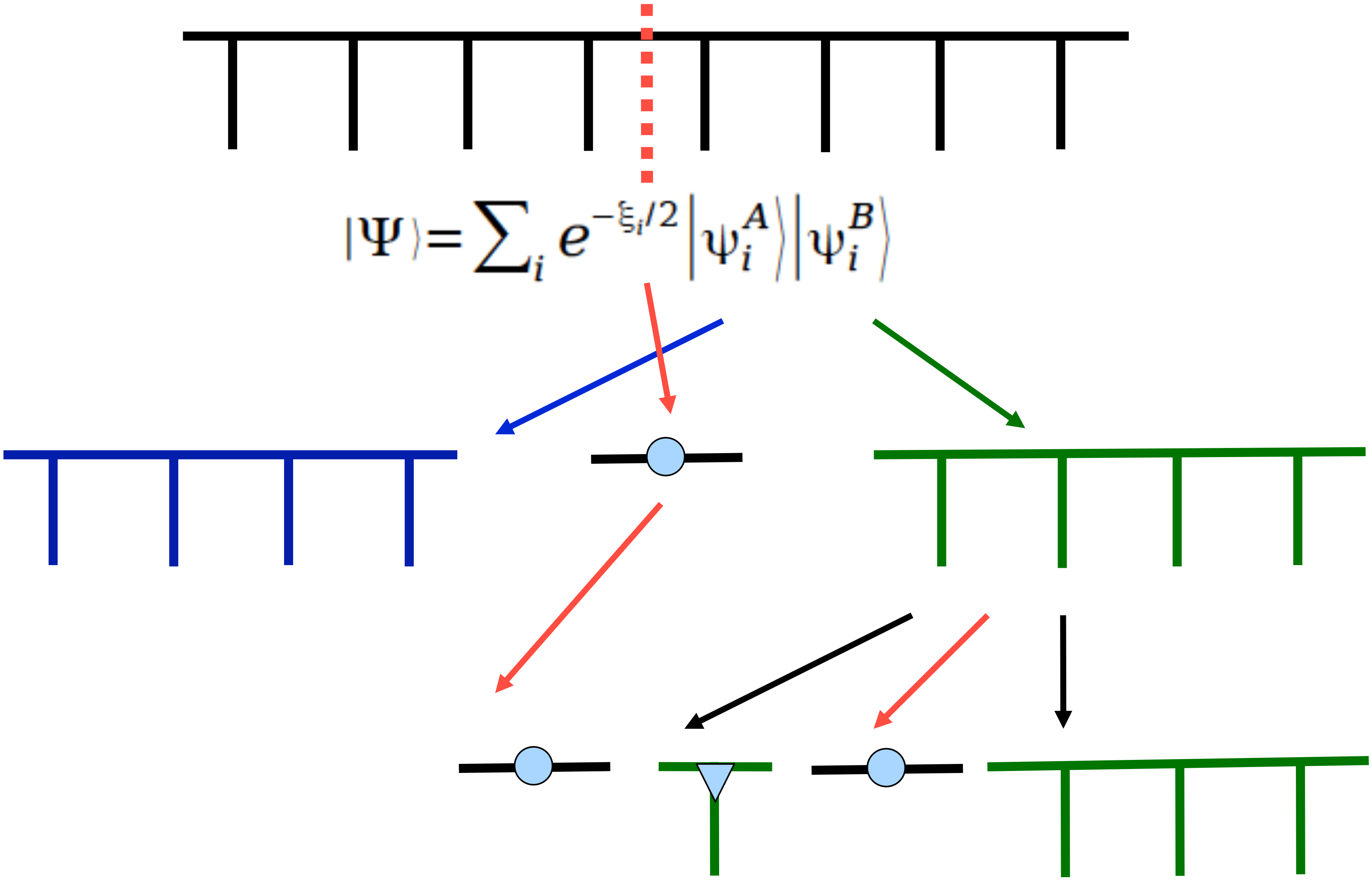
- Trotter-Suzuki decomposition

$$\begin{aligned} H &= H_{\text{odd}} + H_{\text{even}} \\ &= \sum_{n\text{ odd}} h^{[n,n+1]} + \sum_{n\text{ even}} h^{[n,n+1]} \end{aligned} \qquad e^{(A+B)\delta} = e^{A\delta}e^{B\delta} + \mathcal{O}(\delta^2)$$

$$U(\delta t) \approx \left[\prod_{n\text{ odd}} U^{[n,n+1]}(\delta t)\right]\left[\prod_{n\text{ even}} U^{[n,n+1]}(\delta t)\right]$$

$$U^{[n,n+1]}(\delta t) = e^{-i\,\delta t\, h^{[n,n+1]}}$$

Structure of MPS



χ x χ diagonal matrix

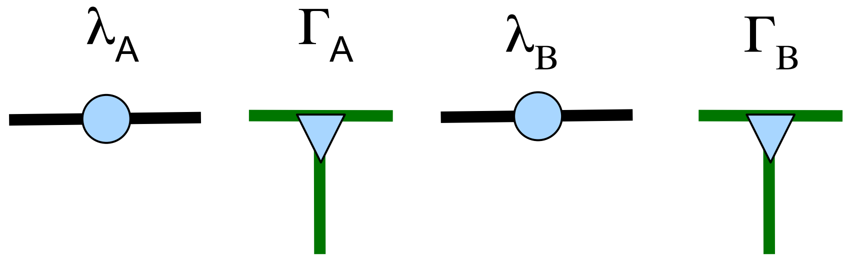
χ x d x χ tensor

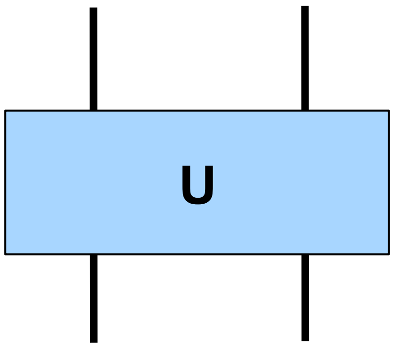
χ x χ diagonal matrix

iTEBD Initialization

Generate a random matrix G (i.e. Γ) with dimension 2 (odd and even sites, A or B) $\times$ d (= 2, spin) $\times \chi \times \chi$ (block dimension). l (i.e. λ) is a diagonal matrix for A and B sites represented by a random vector.

```
import numpy as np
from scipy import integrate

# First define the parameters of the model / simulation
J=1.0; g=0.5; chi=5; d=2; delta=0.005; N=3000;
G = np.random.rand(2,d,chi,chi);l = np.random.rand(2,chi)
```





The Hamiltonian describes a transverse field Ising model on two sites. The diagonal term is $J\sigma_{1z}\sigma_{2z}$, while the off-diagonal terms are $-(g/2)(\sigma_{1x}+\sigma_{2x})$. The basis is $\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\downarrow\rangle\}$.

```
# Generate the two-site time evolution operator
H = np.array( [[J,-g/2,-g/2,0], [-g/2,-J,0,-g/2], [-g/2,0,-J,-g/2], [0,-g/2,-g/2,J]] )
w,v = np.linalg.eig(H)
U = np.reshape(np.dot(np.dot(v,np.diag(np.exp(-delta*(w)))),np.transpose(v)),(2,2,2,2))
```

Construct A Two-Site State

```
# Perform the imaginary time evolution alternating on A and B bonds
for step in range(0, N):
    A = np.mod(step,2); B = np.mod(step+1,2)
```

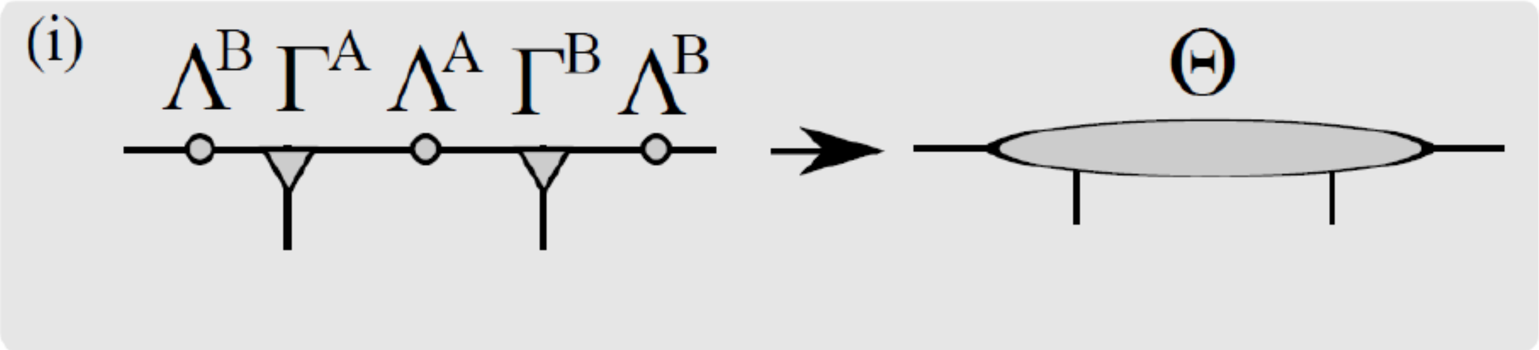


FIG. 1: iTEBD step (i): Construct Θ.

```
# Construct theta
theta = np.tensordot(np.diag(l[B,:]),G[A,:,:,:],axes=(1,1))
theta = np.tensordot(theta,np.diag(l[A,:],0),axes=(2,0))
theta = np.tensordot(theta,G[B,:,:,:],axes=(2,1))
theta = np.tensordot(theta,np.diag(l[B,:],0),axes=(3,0))
```

Two-Site Time Evolution

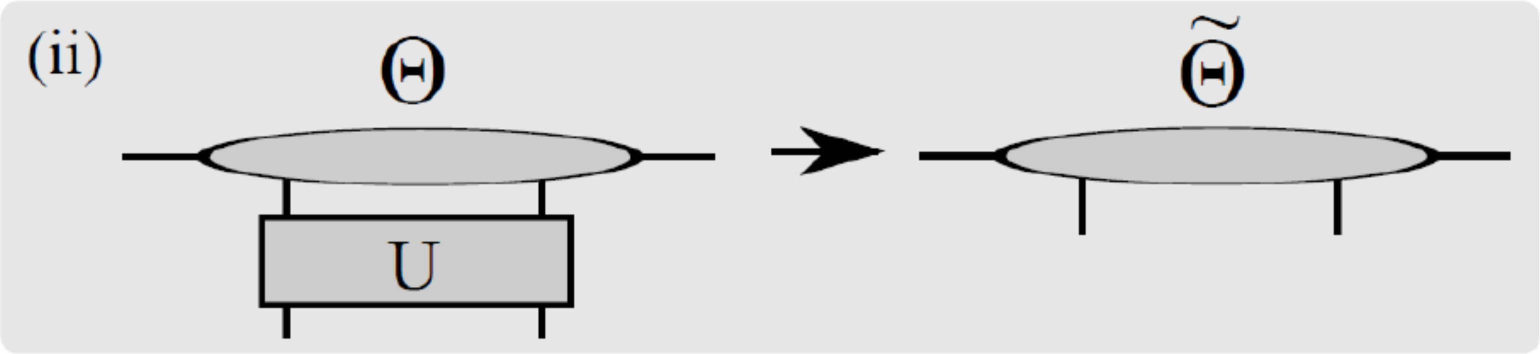


FIG. 2: iTEBD step (ii): Apply time evolution.

```
# Apply U
theta = np.tensordot(theta,U,axes=([1,2],[0,1]));
```

Singular Value Decomposition

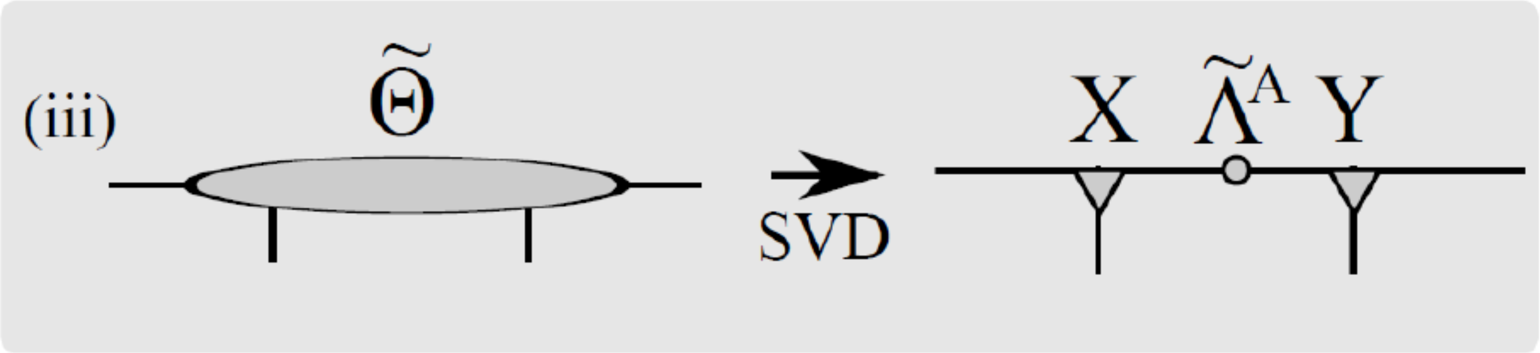


FIG. 3: iTEBD step (iii): Single-value decomposition.

```
# SVD
theta = np.reshape(np.transpose(theta,(2,0,3,1)),(d*chi,d*chi));
X, Y, Z = np.linalg.svd(theta); Z = Z.T
```

Back to the Single-Site Structure

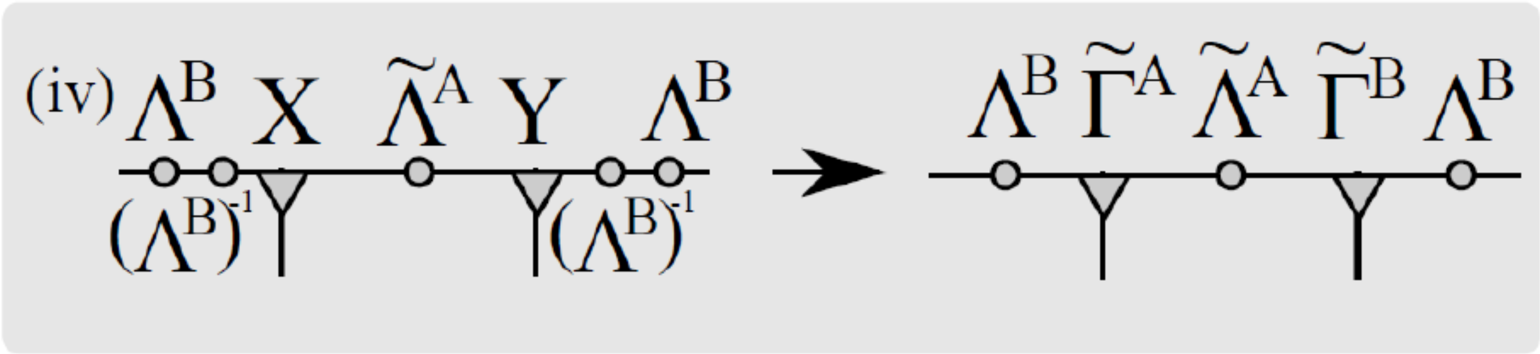
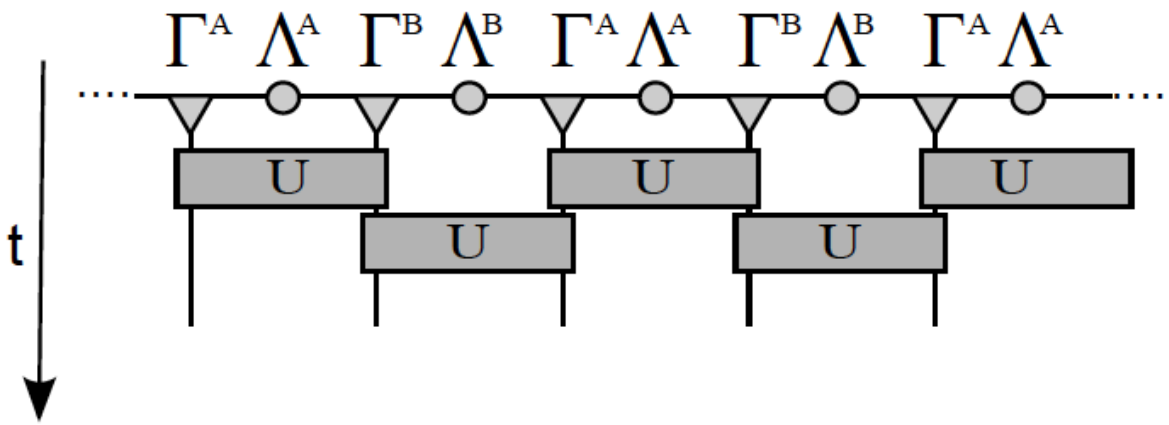


FIG. 4: iTEBD step (iv): Extract the new tensors.

```
# Truncate
l[A,0:chi]=Y[0:chi]/np.sqrt(sum(Y[0:chi]**2))

X=np.reshape(X[0:d*chi,0:chi],(d,chi,chi))
G[A,:,:,:]=np.transpose(np.tensordot(np.diag(l[B,:]**(-1)),X,axes=(1,1)),(1,0,2));

Z=np.transpose(np.reshape(Z[0:d*chi,0:chi],(d,chi,chi)),(0,2,1))
G[B,:,:,:]=np.tensordot(Z,np.diag(l[B,:]**(-1)),axes=(2,0));
```

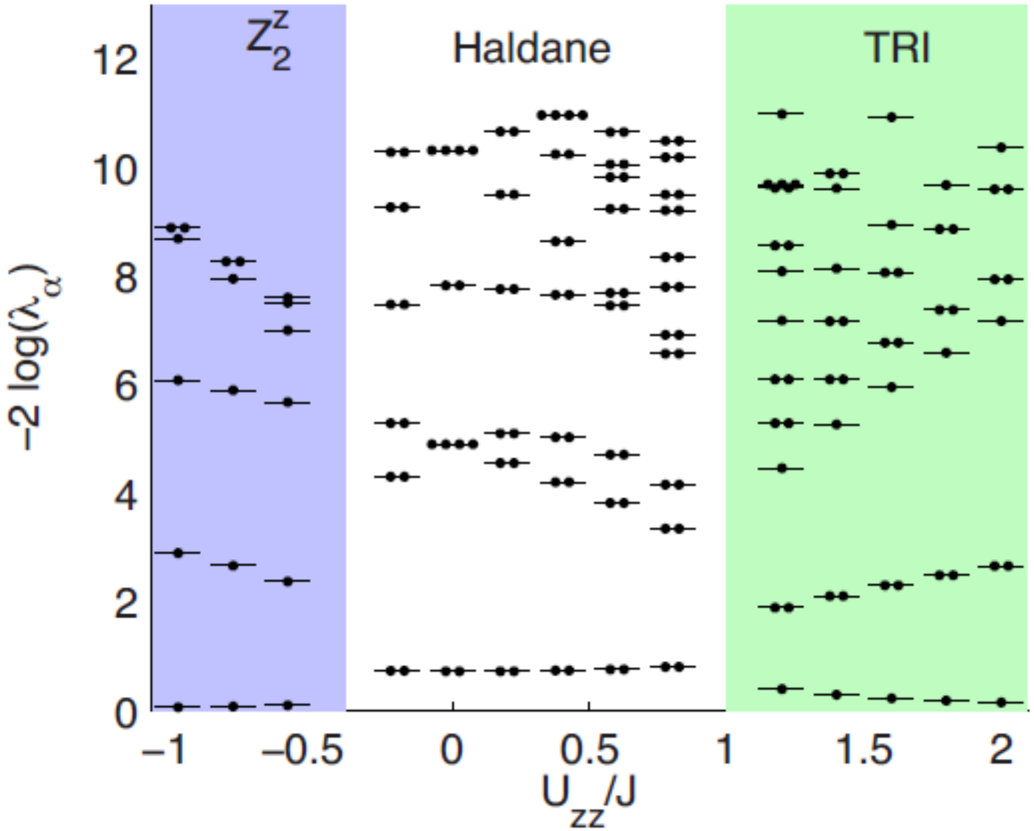
Iteration and Convergence




```
print "E_iTEBD =", -np.log(np.sum(theta**2))/delta/2

f = lambda k,g : -2*np.sqrt(1+g**2-2*g*np.cos(k))/np.pi/2.
E0_exact = integrate.quad(f, 0, np.pi, args=(g,))[0]
print "E_exact =", E0_exact
```

Break and Exercises

- Exercise 1
  - Run the python code and convince yourself that the ground state energy converges to the expected value.
  - Plot the energy of the state during time evolution and observe how it converges.
- Exercise 2
  - Revise the Hamiltonian part to adapt the code for solving the ground state of an Heisenberg spin-1 chain.
  - Calculate the entanglement spectrum for the Heisenberg spin-1 ground state.



$$H_0 = J\sum_j \vec{S}_j \cdot \vec{S}_{j+1} + B_x \sum_j S_j^x + U_{zz} \sum_j (S_j^z)^2$$

Jordan-Wigner Transformation

$$\sigma_j^z = (-1)^{a_j^+ a_j}$$

$\uparrow$ n = 0 $\downarrow$ n = 1

$$a_j = \left(\prod_{k=1}^{j-1} \sigma_k^z\right) \sigma_j^+$$

$$a_j^+ = \left(\prod_{k=1}^{j-1} \sigma_k^z\right) \sigma_j^-$$

$$\left\{a_i, a_j^+\right\} = \delta_{ij}$$

Important to introduce the string, since spins on different sites commute, not anticommute.

$$\sigma_j^x \sigma_{j+1}^x = -\left(a_j - a_j^+\right)\left(a_{j+1} + a_{j+1}^+\right)$$

$$H = J \sum_{j=1}^{N-1} \left(a_j - a_j^+\right)\left(a_{j+1} + a_{j+1}^+\right) - 2h \sum_{j=1}^{N} \left(a_j^+ a_j - \frac{1}{2}\right)$$

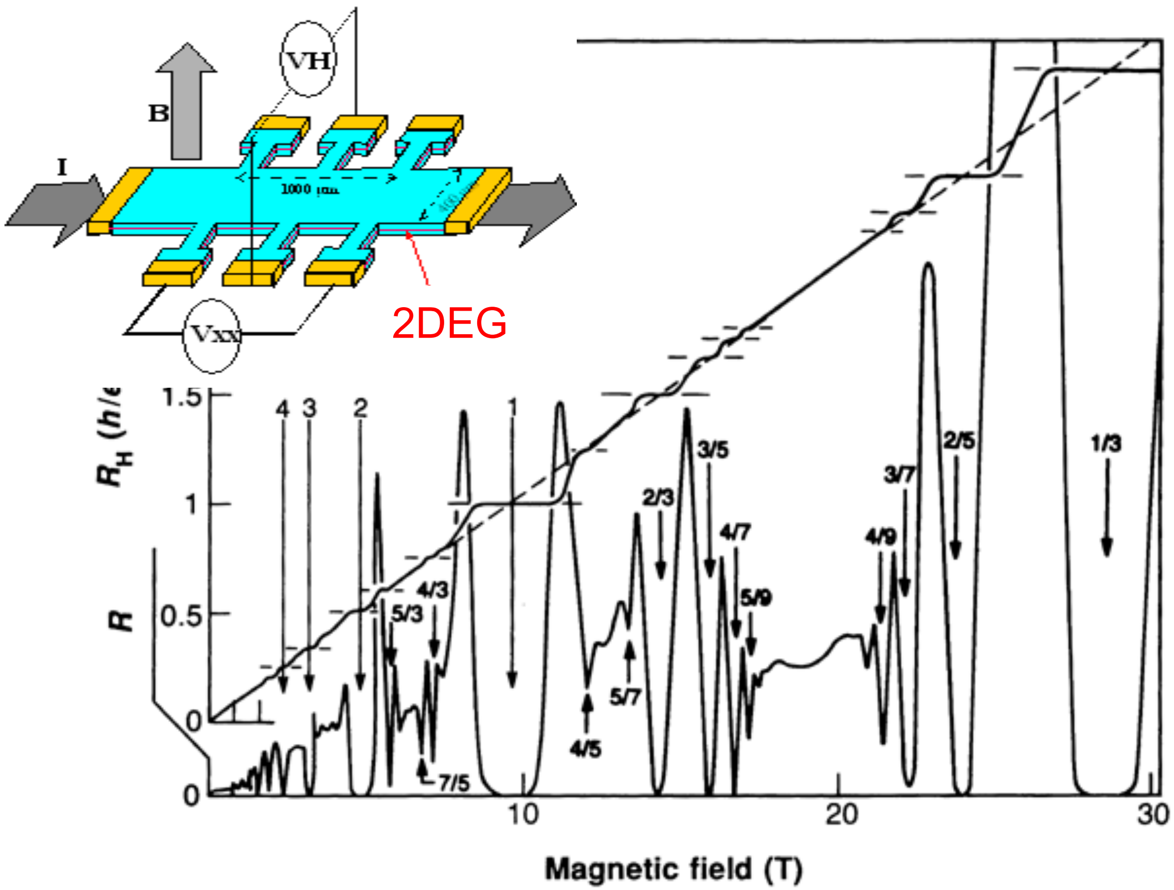
Z_2 symmetry by the fermion parity operator $P_S = (-1)^{\sum_j a_j^+ a_j}$

After the Break, …

- Discussion on *p*-wave superconductors
  - Connection to the Moore-Read state
  - Construction of artificial 1D *p*-wave superconductors and experimental signatures
- Laughlin state in the era of MPS
- Subscribe PhysicsChat
  - Type “Nanjing2016”

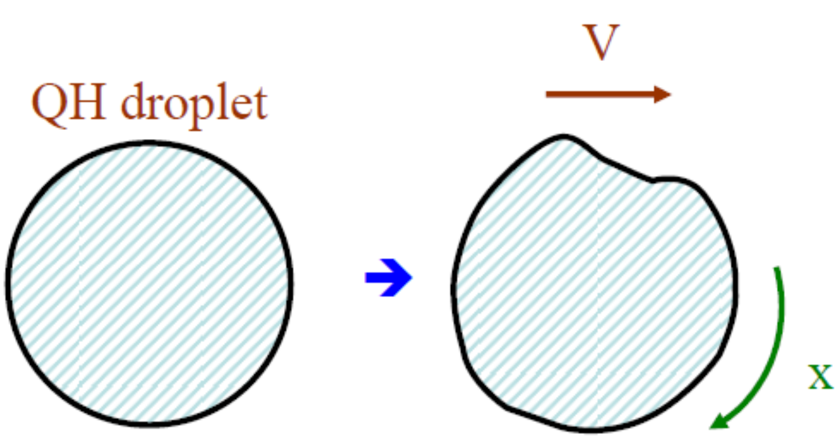
http://zimp.zju.edu.cn/~xinwan/Nanjing2016/ (available after G20)

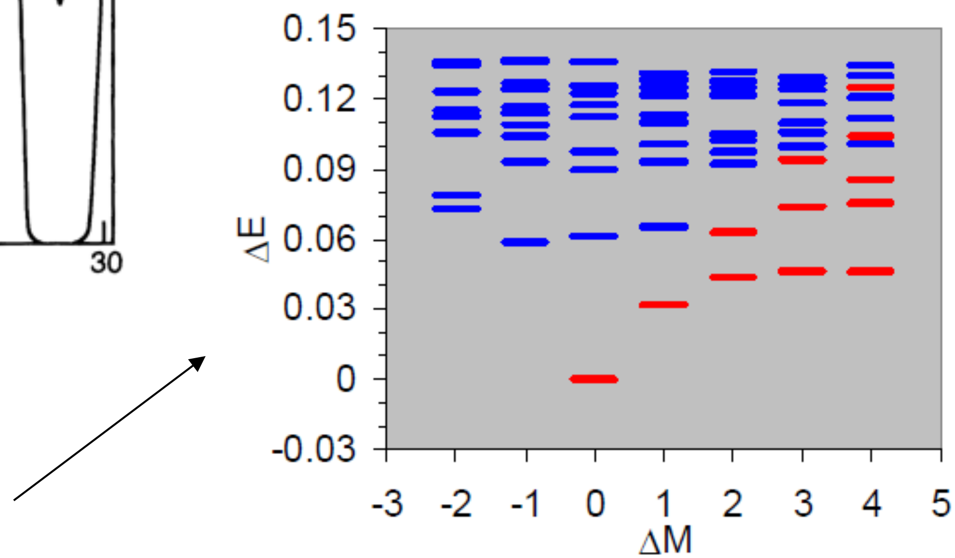
Fractional Quantum Hall Effect



$$R_H = \frac{V_H}{I} = \frac{h}{\nu e^2}$$

$$R = \frac{V_{xx}}{I}$$



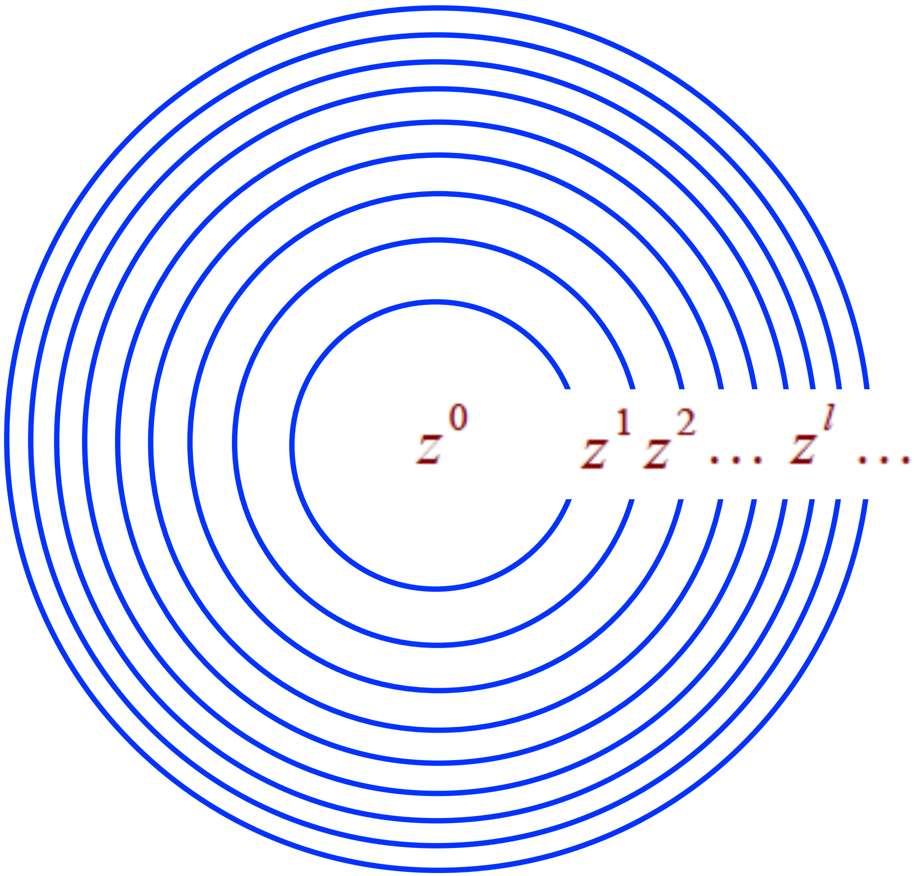


Laughlin State – Disk Geometry

- In the LLL, electron-electron interaction is not a perturbation.

$$\phi_l(z) \sim z^l e^{-|z|^2/4}$$

$$z = x + iy$$



- Basic requirement for an electron wave function in the LLL:
  - antisymmetric function
  - analytic function
  - a universal Gaussian factor

- Laughlin state

$$\Psi_L = \prod_{i<j} (z_i - z_j)^m e^{-\sum_i |z_i|^2/4}$$

$$R_l = \sqrt{\langle l | r^2 | l \rangle} = \sqrt{2(l+1)}$$

Model Hamiltonian for the Laughlin State

- Laughlin wavefunction is the ground state of

$$H_{hardcore} = \sum_{i<j}^{N} \partial_i^{q-1} \delta^2 (z_i - z_j)$$

$$\Psi_L = \prod_{i<j} (z_i - z_j)^3 e^{-\sum_i |z_i|^2/4}$$

- Its LLL projection has a simple pseudopotential form
  - Two-particle wavefunction

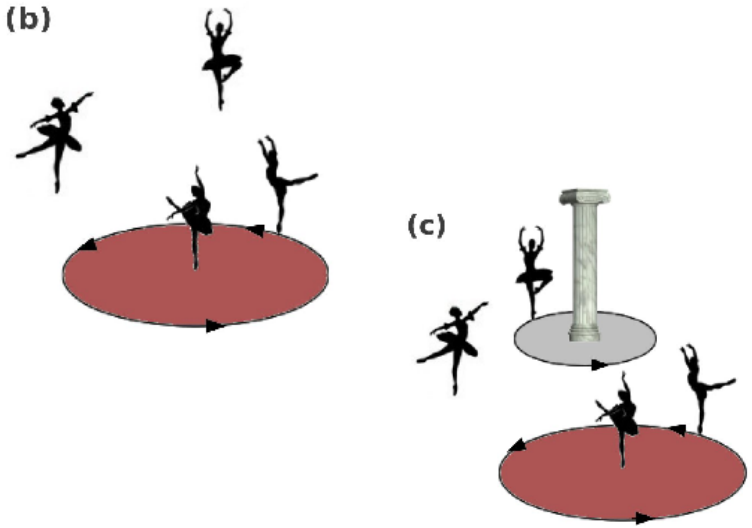
$$(z_1 + z_2)^M (z_1 - z_2)^m$$

  - Interaction can be written, in general, as

$$H_i = \sum_m V_m P_m(1,2)$$

  - One produces the 1/3 Laughlin factor by $V_1 > 0$ only
  - Charge *ne/q* quasiparticles can be introduced by inserting fluxes

When N = 2 particles approach the same point, the wavefunction vanishes as $q = 3$ powers.



Plasma Analogy

- How are particle distributed?

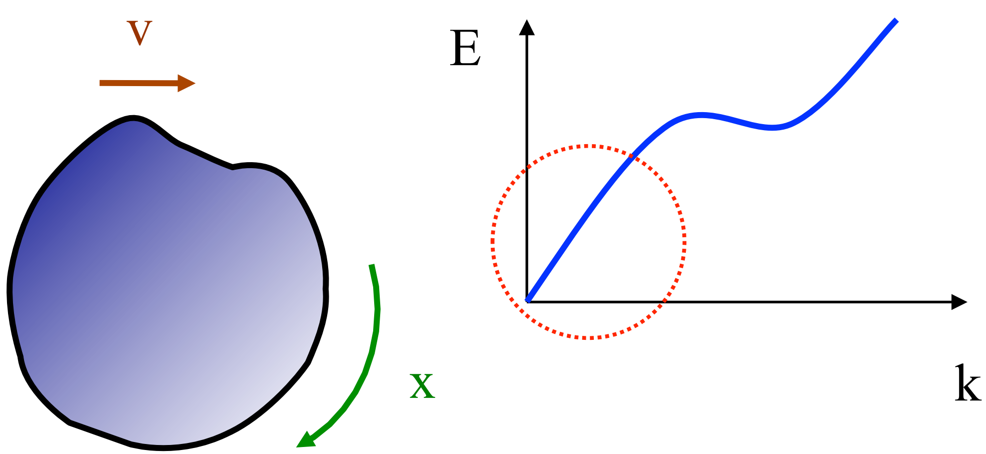
$$|\Psi_{Laughlin}|^2 = e^{-\beta_{plasma} H_{plasma}}$$

- Boltzmann weight of a 2D plasma (classical charged particles) at temperature T = 1/(2q) with uniform background charge.

$$H_{plasma} = -\sum_{i<j} \ln|z_i - z_j| + \frac{1}{4q} \sum_i |z_i|^2$$

- The 2D plasma is in a screened phase at high T (q < 70), so the Laughlin wavefunction describes a disk of fluid with uniform density.

- With a bulk gap, the Laughlin state describes an incompressible quantum liquid. But the edge is gapless and supports a branch of chiral boson modes, which describes the deformation of the charge density at the edge.



Chiral Luttinger liquid

Decoding the Laughlin State

- Laughlin state at 1/3 filling

$$\Psi_L = \prod_{i<j} (z_i - z_j)^3 e^{-\sum_i z_i^2/4}$$

- A two-particle example

$$(z_1 - z_2)^3 = 1 \cdot (z_1^3 - z_2^3) + (-3) \cdot (z_1^2 z_2 - z_1 z_2^2)$$

$$= -\begin{vmatrix} 1 & 1 \\ z_1^3 & z_2^3 \end{vmatrix} + 3 \begin{vmatrix} z_1 & z_2 \\ z_1^2 & z_2^2 \end{vmatrix} = -sl_{[3,0]} + 3\, sl_{[2,1]}$$

1 ●□□● + (- 3) □●●□

Orbitals: 0,1,2,3

Root configuration

squeeze

Few-Particle Wave Functions

N = 2

```
|0110)  (2,1)  ---> -3
|1001)  (3,0)  ---> 1
```

A complete set of basis can be generated by squeezing the root configuration

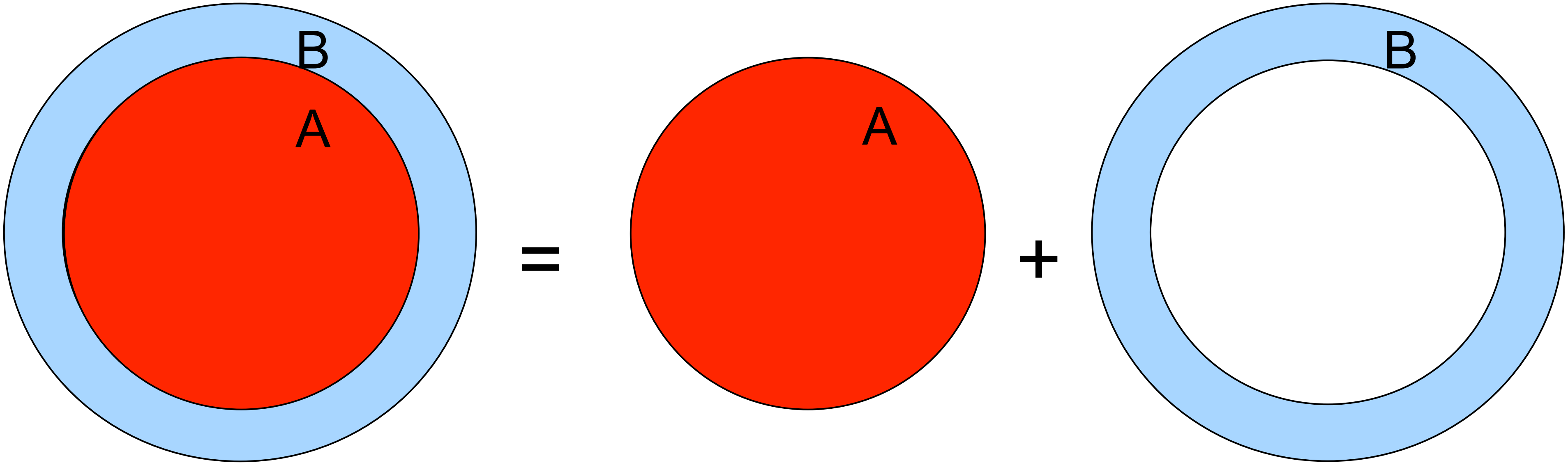
N = 3

```
|0011100)  (4,3,2)  ---> -15
|0101010)  (5,3,1)  ---> 6
|1000110)  (5,4,0)  ---> -3
|0110001)  (6,2,1)  ---> -3
|1001001)  (6,3,0)  ---> 1
```

N = 4

```
|0001111000)  (6,5,4,3)  ---> 105
|0010110100)  (7,5,4,2)  ---> -45
|0011001100)  (7,6,3,2)  ---> -6
|0100101100)  (7,6,4,1)  ---> 27
|1000011100)  (7,6,5,0)  ---> -15
|0011010010)  (8,5,3,2)  ---> 27
|0100110010)  (8,5,4,1)  ---> -9
|0101001010)  (8,6,3,1)  ---> -12
|1000101010)  (8,6,4,0)  ---> 6
|0110000110)  (8,7,2,1)  ---> 9
|1001000110)  (8,7,3,0)  ---> -3
|0011100001)  (9,4,3,2)  ---> -15
|0101010001)  (9,5,3,1)  ---> 6
|1000110001)  (9,5,4,0)  ---> -3
|0110001001)  (9,6,2,1)  ---> -3
|1001001001)  (9,6,3,0)  ---> 1
```

Entanglement Entropy



$$|\Psi\rangle = \sum_i e^{-\xi_i/2} \left|\Psi_i^A\right\rangle \otimes \left|\Psi_i^B\right\rangle \qquad \rho_A = Tr_B\left(|\Psi\rangle\langle\Psi|\right) = e^{-H_E}$$

Entanglement entropy:

$$\sum_i \xi_i e^{-\xi_i} = -Tr_A\left(\rho_A \ln \rho_A\right)$$
$$= \alpha L - \gamma + O(1/L)$$

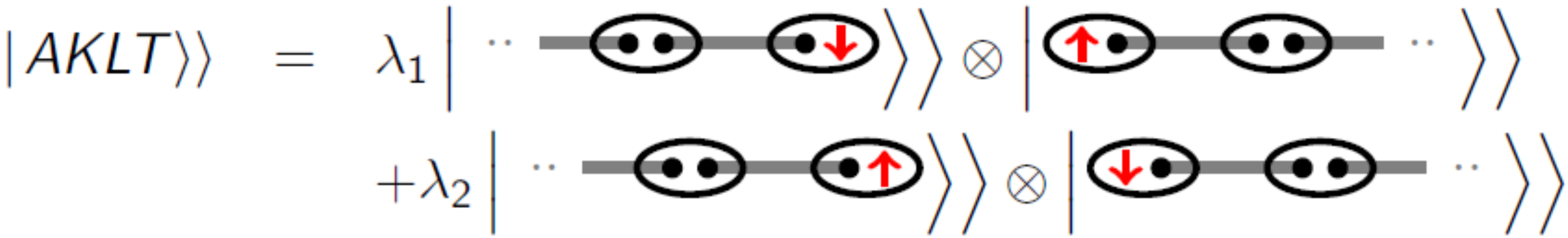
Entanglement spectrum: ξ_i

Kitaev & Preskill (2006); Levin & Wen (2006); Li & Haldane (2008)

Entanglement Spectrum for N = 4

```
# Entanglement spectrum:
# nel M      E_entanglement       DM_eigenvalue
ES: 1 0      1.731301500191       0.177053824363
ES: 2 3      2.400732154134       0.090651558074
```
~~ES: 2 3 73.473601139000 0.000000000000~~
```
ES: 2 4      2.082278423015       0.124645892351
```
~~ES: 2 4 73.473601139000 0.000000000000~~
```
ES: 2 5      1.798045305178       0.165622313098
ES: 2 5      3.091674050844       0.045425845542
ES: 2 6      2.242127123957       0.106232294618
ES: 2 7      1.394829263570       0.247875354108
ES: 3 9      3.158417855831       0.042492917847

ES: 2 3      2.400732154134       0.090651558074
|01100>        -0.866025403784
|10010>         0.500000000000
```

Entanglement Connection btw AKLT & FQHE



$$\left\|\Psi_{Laughlin}(N)\right\rangle\rangle = \lambda_1 \left\|\Psi_{Laughlin}(N/2)\right\rangle\rangle \otimes \left\|\Psi_{Laughlin}(N/2)\right\rangle\rangle$$

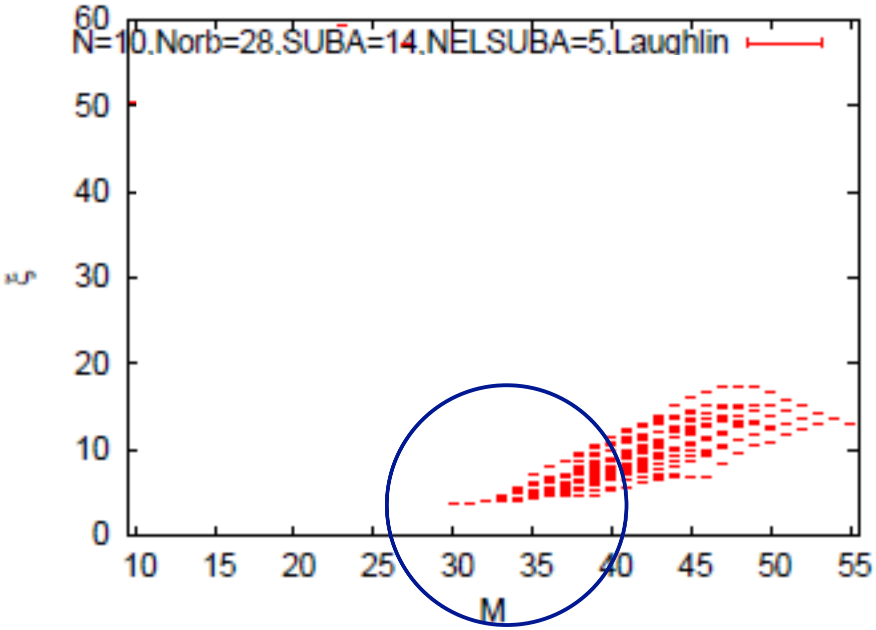
$$+ \quad \lambda_2 \left\|\Psi_{\Delta M=1}(N/2)\right\rangle\rangle \otimes \left\|\Psi_{\Delta M=1}(N/2)\right\rangle\rangle$$

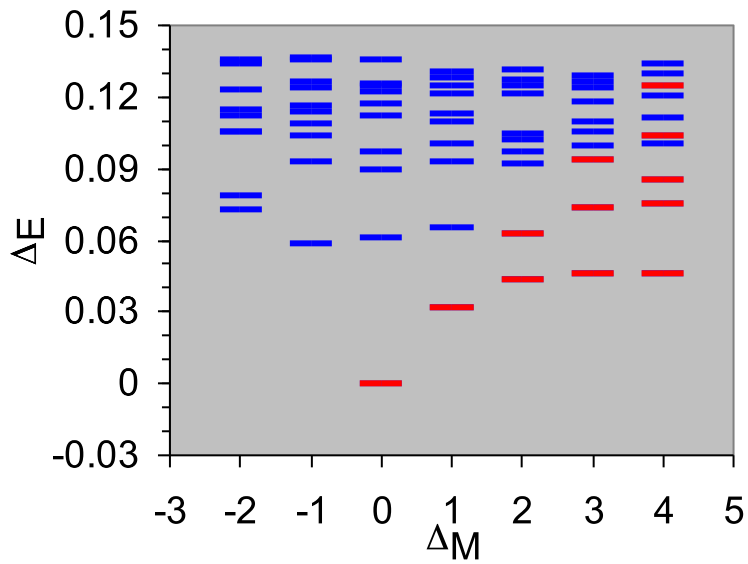
$$+ \quad \cdots$$

For a path-integral formalism, see Dubail, Rezayi & Read, PRB (2013)

Entanglement Spectrum vs Edge Spectrum

$$|\Psi\rangle = \sum_i e^{-\xi_i/2} |\Psi_i^A\rangle \otimes |\Psi_i^B\rangle$$





XW et al. (2003)

$$\Delta M = M_0 + \Delta M(\{n_l\}) = M_0 + \sum_l n_l l$$

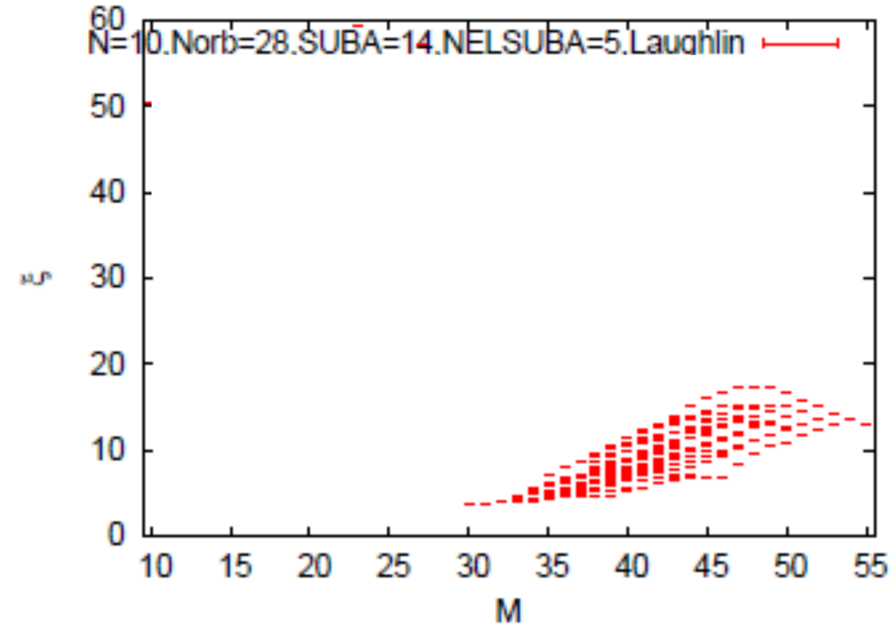
$$\Delta E = E_0 + \Delta E(\{n_l\}) = E_0 + \sum_l n_l \epsilon_l$$

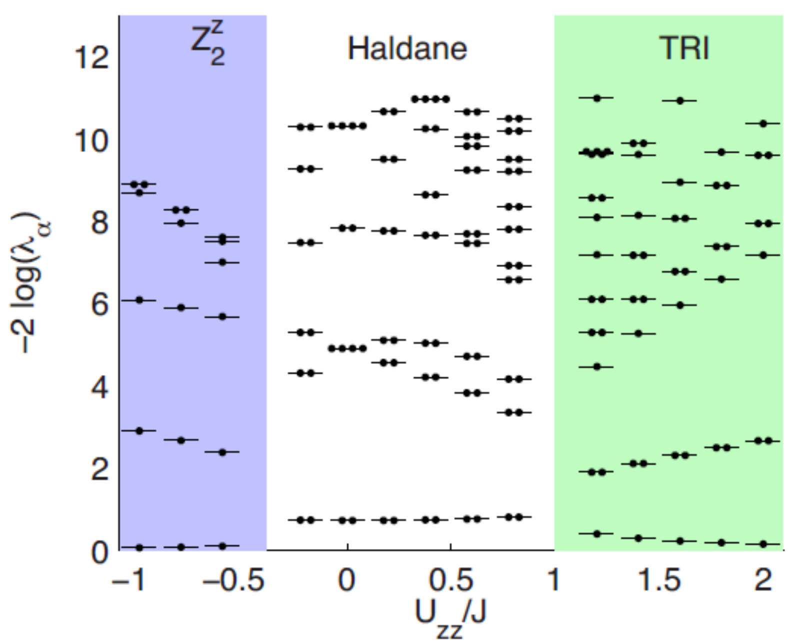
The entanglement Hamiltonian $H_E = -\ln \rho_A$ is isospectral to an operator that is local along the cut between A and B.

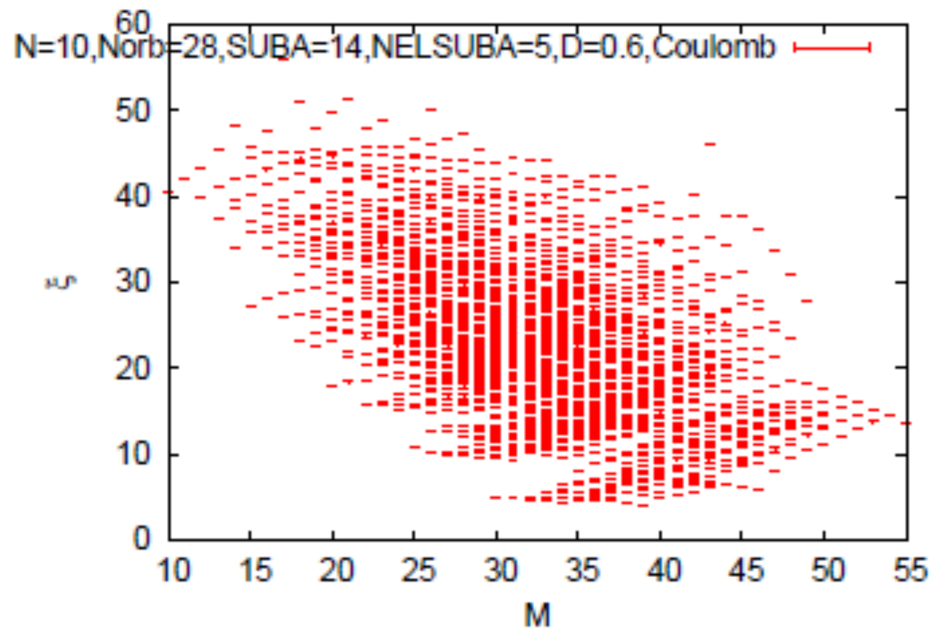
Dubail, Read & Rezayi (2013)

From Ideal to Realistic Spectra

ln 2 







Conformal Field Theory (CFT) Language

- Laughlin state $\nu = 1/m$ <==> chiral boson CFT ($c = 1$)
- Quasiparticles $\phi_0 = 1$, ϕ_1, ϕ_2, ..., ϕ_{m-1} <===> primary fields
  - The primary fields of the CFT are in one-to-one correspondence with the quasiparticles in the topological phase, with electric charge 0, e/m, 2e/m, ….
  - Fusion rules (Abelian)

$$\varphi_{r_1} \times \varphi_{r_2} = \varphi_{r_1 + r_2} \qquad d = 1$$

  - Total quantum dimension $D = sqrt(m)$, encoded in the subleading size- and shape-independent term in the entanglement entropy.

$$S_A = \alpha L - \ln D + \cdots$$

- In the FQH regime, CFT now becomes a standard way to guess new (2+0)d wavefunctions, or to study (1+1)d quasiparticle dynamics.

Laughlin Wavefunction from CFT (ν = 1/m)

- Electron operator is a vertex operator

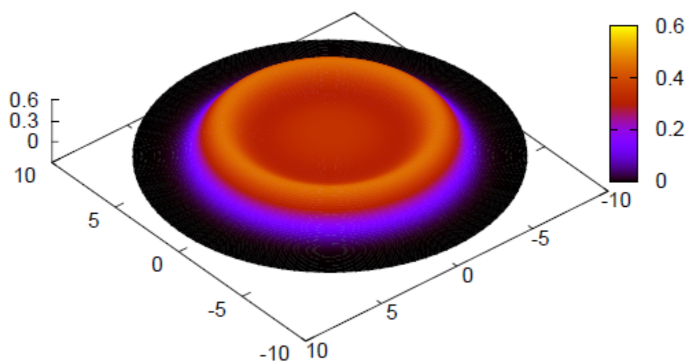
$$\psi_e(z) \equiv V_\alpha(z) = :e^{i\alpha\phi(z)}:, \alpha = \sqrt{m}$$

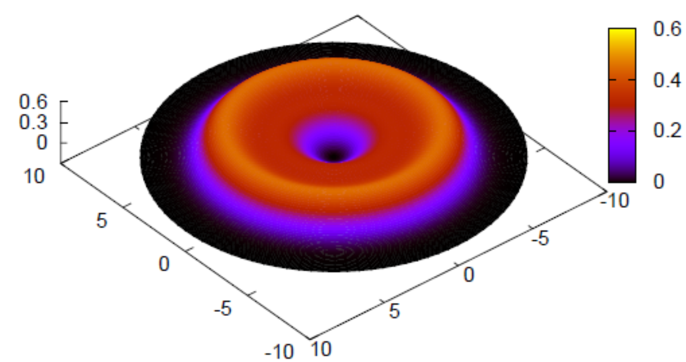
  - *f* (z) is a free massless chiral bosonic field $\langle \phi(z)\phi(w)\rangle = -\ln(z-w)$

$$OPE \quad :e^{i\sqrt{m}\varphi(z)}::e^{i\sqrt{m}\varphi(w)}: = (z-w)^m :e^{i\sqrt{m}[\varphi(z)+\varphi(w)]}:$$

- Laughlin wavefunction

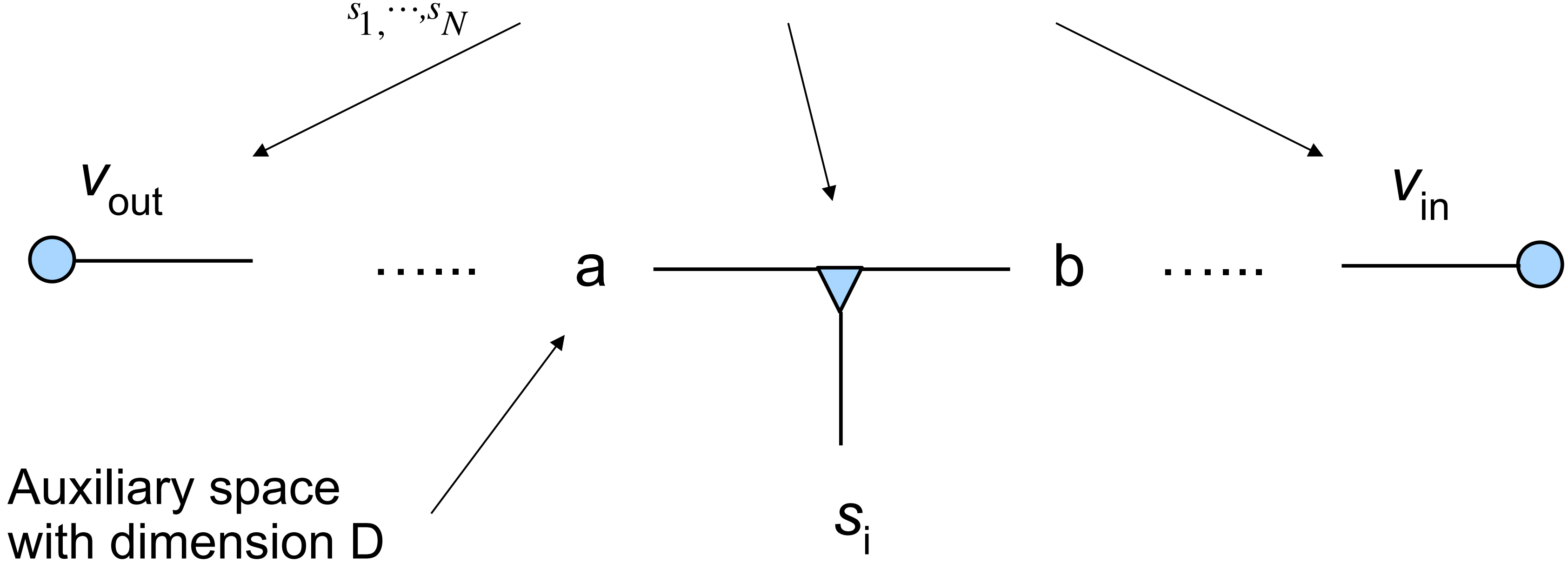
$$\langle mN | V_{\sqrt{m}}(z_1)\cdots V_{\sqrt{m}}(z_N)|0\rangle = \prod_{i<j}(z_i - z_j)^m e^{-\sum_i |z_i|^2/4}$$

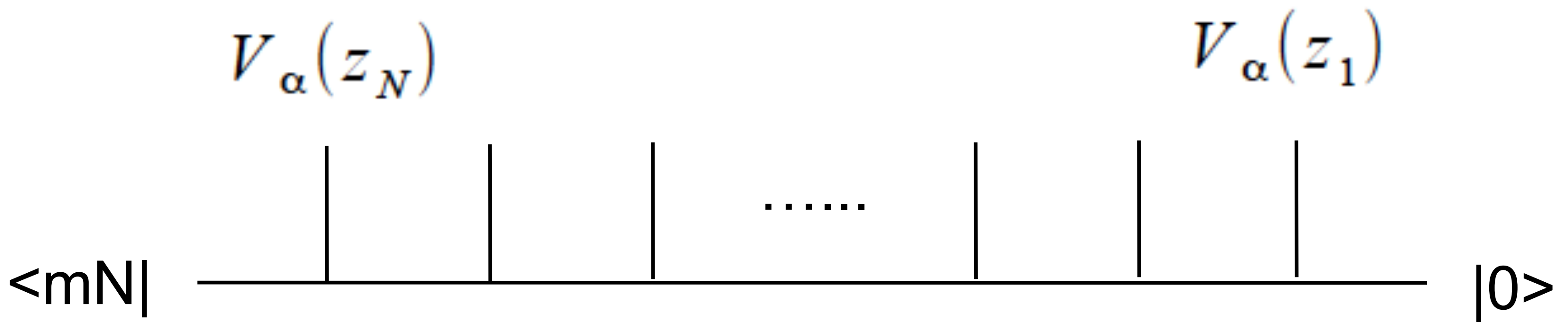




Matrix Product States & Conformal Blocks

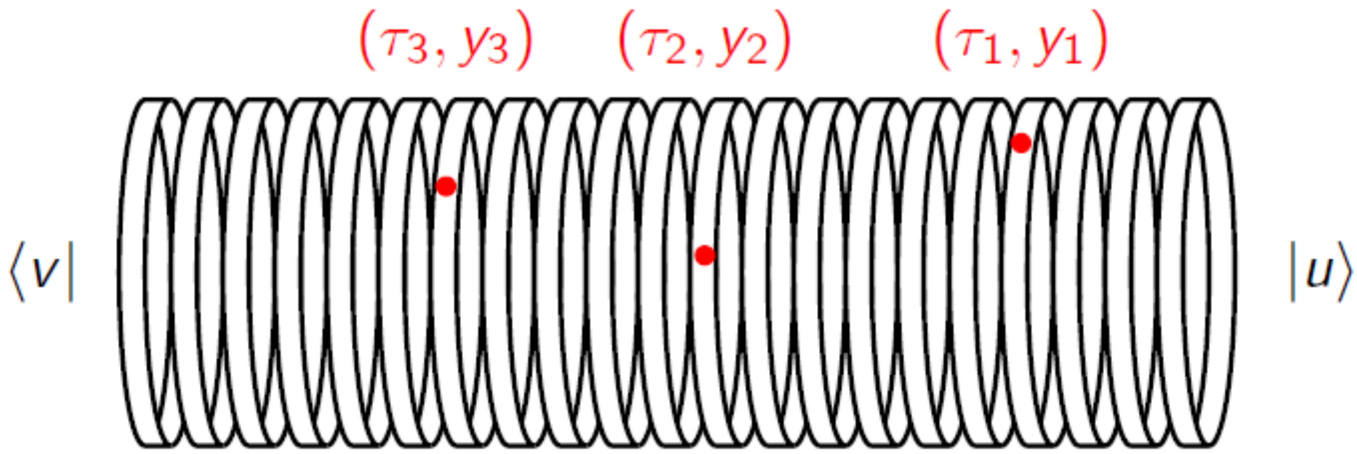
$$|\Psi\rangle = \sum_{s_1,\cdots,s_N} \langle v_{out}|A(s_N)\cdots A(s_1)|v_{in}\rangle |s_1 s_2 \cdots s_N\rangle$$





$$|\Psi\rangle = \langle mN|V_{\sqrt{m}}(z_1)\cdots V_{\sqrt{m}}(z_N)|0\rangle$$

Cylinder Geometry

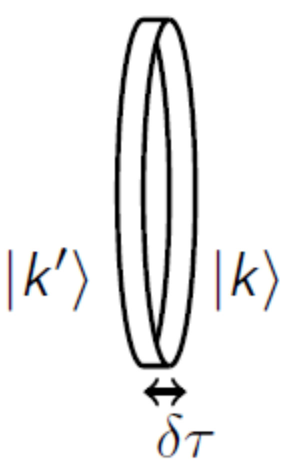


$$\langle v|\, e^{-(\tau_4-\tau_3)\hat{H}} V(y_3) e^{-(\tau_3-\tau_2)\hat{H}} V(y_2) e^{-(\tau_2-\tau_1)\hat{H}} V(y_1) e^{-\tau_1 \hat{H}}\, |u\rangle$$

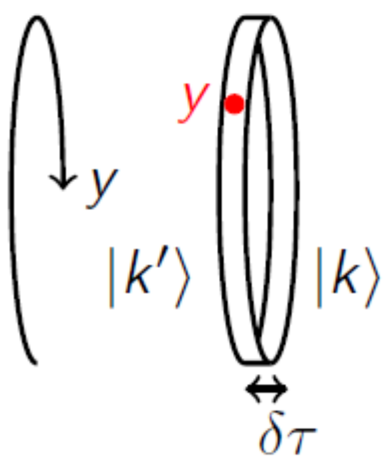
$$z_j = \tau_j + iy_j$$

$$\langle k'|\, e^{-\delta\tau\ \hat{H}}\, |k\rangle \qquad\qquad \langle k'|\, V(y)\, |k\rangle$$

B^0 =



B^1 =



What is the Auxiliary Space?

- In the CFT space, a vertex operator corresponds to a *primary* state

  $|N\rangle = e^{iN\phi_0/R}|0\rangle$ with a U(1) charge Q = N/R $R=\sqrt{1/\nu}=1/\beta$

  $$a_0|N\rangle = \frac{N}{R}|N\rangle \qquad a_n|N\rangle = 0 \quad \text{for } n > 0$$

- The rest generated by repeatedly applying a_{-n} (similar to creation ops.)

  $$|N,\mu\rangle = \frac{1}{\sqrt{z_\mu}}\prod_{i=1}^{\infty} a_{-i}^{m_i}|N\rangle \qquad z_\mu = \prod_i i^{m_i} m_i! \qquad |\mu| = \sum_i m_i i$$

- Hilbert space consists of all $|N,\mu\rangle$

  $$|h_N\rangle = span\{a_{-n_1}a_{-n_2}\cdots|0\rangle\}$$

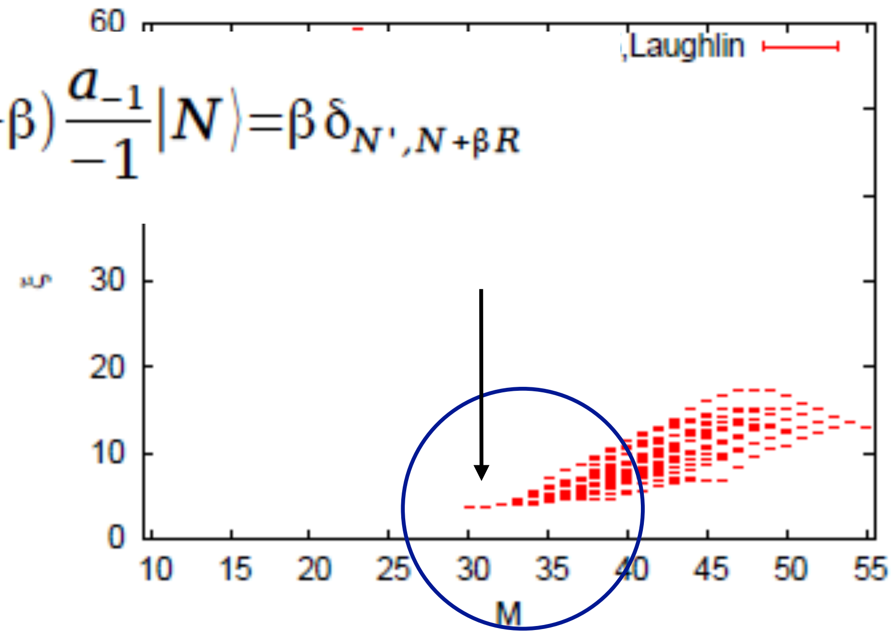
- Note: mode expansion of chiral boson

  $$\phi(z) = \phi_0 - i a_0 \log(z) + i\sum_{n\neq 0}^{\infty}\frac{1}{n}a_n z^{-n} \qquad [\phi_0, a_0] = i, \quad [a_n, a_m] = n\delta_{n+m,0}$$

Matrix Elements – with Examples

$$\langle N',\{\}|e^{i\beta\phi(1)}|N,\{1\}\rangle=\langle N'|e^{i\beta\phi_0}(-\beta)\frac{a_1}{1}a_{-1}|N\rangle=-\beta\delta_{N',N+\beta R}$$

$$\langle N',\{1\}|e^{i\beta\phi(1)}|N,\{\}\rangle=\langle N'|a_1 e^{i\beta\phi_0}(-\beta)\frac{a_{-1}}{-1}|N\rangle=\beta\delta_{N',N+\beta R}$$



Generally, $$|N,\mu\rangle=\frac{1}{\sqrt{Z_\mu}}\prod_{i=1}^{\infty}a_{-i}^{m_i}|N\rangle$$

$$\langle N',\mu'|:e^{i\beta\phi(z)}:|N,\mu\rangle=z^{\beta N/R+|\mu'|-|\mu|}A^{(\beta)}_{\mu',\mu}\delta_{N',N+\beta R}$$

$$A^{(\beta)}_{\mu',\mu} = \prod_{i=1}^{\infty}\sum_{r=0}^{m'_j}\sum_{s=0}^{m_j}\delta_{m'_j+s,m_j+r}\frac{(-1)^s}{\sqrt{r!s!}}\left(\frac{\beta}{\sqrt{j}}\right)^{r+s}\sqrt{\binom{m'_j}{r}\binom{m_j}{s}}$$

Site-Independent Matrices

- *n*th orbital with symmetric background charge

$$B^m = e^{-i\sqrt{\nu}\phi_0/2} \frac{V_{-h}^m}{\sqrt{m!}} e^{-i\sqrt{\nu}\phi_0/2} \qquad V(z) = \sum_n z^n V_{-n-h}$$

*n*th LLL orbital

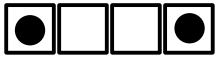
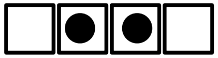
- Explicitly, the matrix elements are

$$\langle N', \mu' | B^0 | N, \mu | = \delta_{N', N-1} \delta_{\mu', \mu}$$

$$\langle N', \mu' | B^1 | N, \mu | = \delta_{N', N+q-1} \delta_{\mu' + (N+N')/2, \mu} A_{\mu', \mu}^{(\beta)}$$

- Normalization on a specific geometry can be taken care separately.

Laughlin State, Once Again

1  + (- 3) 

Orbitals: 0,1,2,3

m		0		1		0		0		1		0		
N	0		1		-1		0		1		-1		0	
μ	0		0		0		0		0		0		0	

1 * 1 * 1 * 1 = 1

m		0		0		1		1		0		0		
N	0		1		2		0		-2		-1		0	
μ	0		0		0		1		0		0		0	

1 * (- b) * b * 1 = - 3

Example Matrices & Results

Pmax = 0
Dim (chi, chi_n, chi_p) = 3 3 1

```
B[0] = [ [ 0.  1.  0.]
         [ 0.  0.  1.]
         [ 0.  0.  0.] ]
B[1] = [ [ 0.  0.  0.]
         [ 0.  0.  0.]
         [ 1.  0.  0.] ]
```

[1001001001] 1

Squeeze operation

$$\cdots 00100 \cdots 00100 \cdots$$

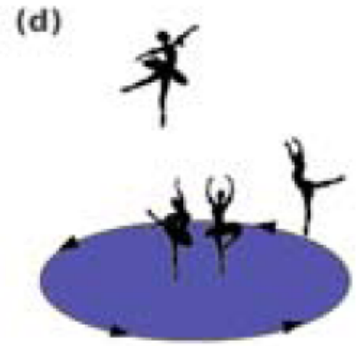
$$\cdots 00010 \cdots 01000 \cdots$$

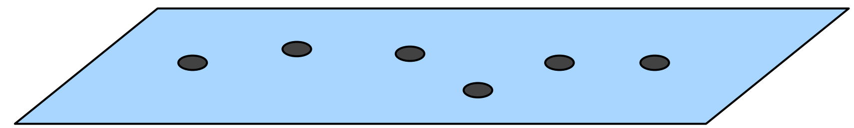
Pmax =	1	2	3	4
Dim =	10 5 2	28 7 4	63 9 7	132 11 12
[0001111000]	0.0	0.0	0.0	105.0
[0010110100]	0.0	0.0	-45.0	-45.0
[0011001100]	0.0	-6.0	-6.0	-6.0
[0100101100]	0.0	27.0	27.0	27.0
[1000011100]	0.0	-15.0	-15.0	-15.0
[0011010010]	0.0	27.0	27.0	27.0
[0100110010]	0.0	-9.0	-9.0	-9.0
[0101001010]	-12.0	-12.0	-12.	-12.0
[1000101010]	6.0	6.0	6.0	6.0
[0110000110]	9.0	9.0	9.0	9.0
[1001000110]	-3.0	-3.0	-3.0	-3.0
[0011100001]	0.0	-15.0	-15.0	-15.0
[0101010001]	6.0	6.0	6.0	6.0
[1000110001]	-3.0	-3.0	-3.0	-3.0
[0110001001]	-3.0	-3.0	-3.0	-3.0
[1001001001]	1.0	1.0	1.0	1.0

Color-coded based on level of squeezing

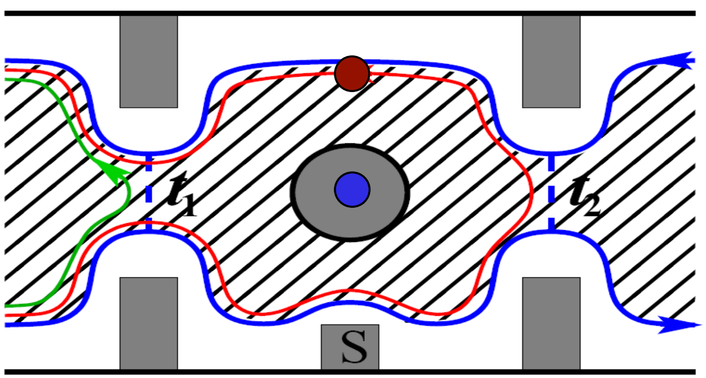
从分数量子霍尔效应到拓扑量子计算

二维电子在极低温和强磁场下，可以跳一种复杂的舞蹈，我们说形成一种奇异的量子液体。即分数量子霍尔态。



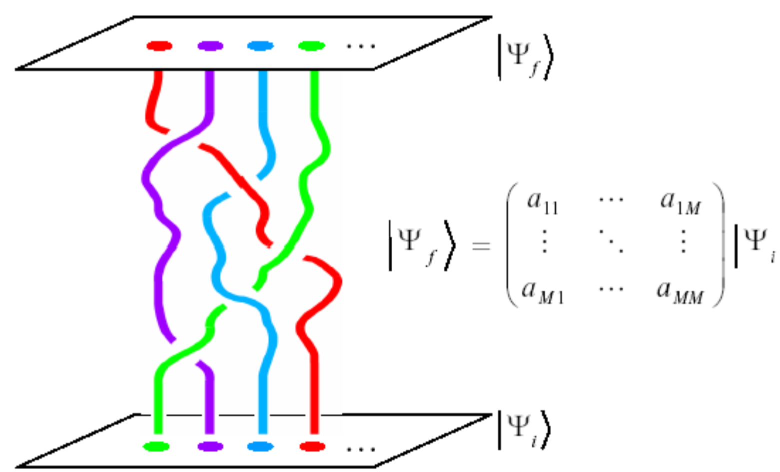


这样的二维系统可以激发出满足奇异的非阿贝尔统计性质的准粒子。当准粒子相互交换时，系统的基态在一个受拓扑保护的简并空间变换。



非阿贝尔准粒子的性质可以用双点接触干涉实验来验证，是目前实验研究的焦点。贝尔实验室的实验结果与理论期待有相当大的符合性。

非阿贝尔准粒子的在2+1维时空的演化，实现了拓扑量子计算。这种计算不受居域的环境噪声的微扰，具有很好的量子相干性，



量子门的构造就是要找到2+1维时空中的一个辫子，对应着态空间的一个幺正变换。

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